Mathematics 1101Y – Calculus I: Functions and calculus of one variable

Trent University, 2011–2012

Quizzes

Quiz #1. Monday, 19 September, 2011. *[10 minutes]*

1. Find the intercepts of the parabola $y = x^2 - 2x - 3$, and sketch its graph. *[5]*

Solution. The y-intercept is obtained by plugging $x = 0$ into the equation of parabola. Since $y = 0^2 - 2 \cdot 0 - 3 = -3$, the parabola meets the y-axis at the point $(0, -3)$.

The x-intercepts are the values of x for which $y = x^2 - 2x - 3 = 0$; we find these with the help of the quadratic formula:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1} = \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2} = 1 \pm 2 = \begin{cases} 1 - 2 = -1 \\ 1 + 2 = 3 \end{cases}$$

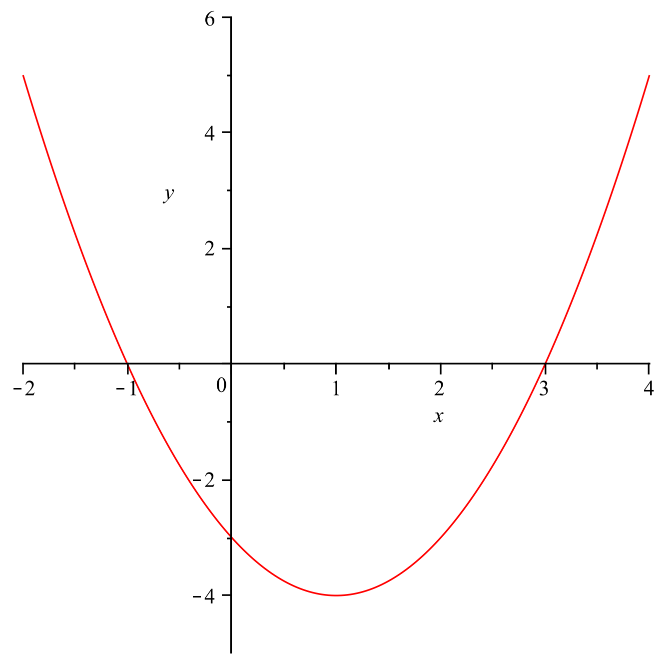
It follows that the parabola meets the x-axis at the points $(-1, 0)$ and $(2, 0)$.

Alternatively, one could find the x-intercepts by factoring the quadratic expression $x^2 - 2x - 3$ in some way. Since $x^2 - 2x - 3 = (x + 1)(x - 3)$, we get zero at $x = -1$ and $x = 3$, respectively.

The intercepts obtained above and the knowledge that the parabola opens upward because x^2 has a positive coefficient is enough for a crude sketch of the parabola. (One could plot a few more points easily enough, too.) We cheat slightly and use `Maple` – the old-worksheet-style command

```
> plot(x^2-2*x-3,x=-2..4,y=-5..6);
```

generates the following graph:



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Quiz #2. Monday, 26 September, 2011. *[10 minutes]*

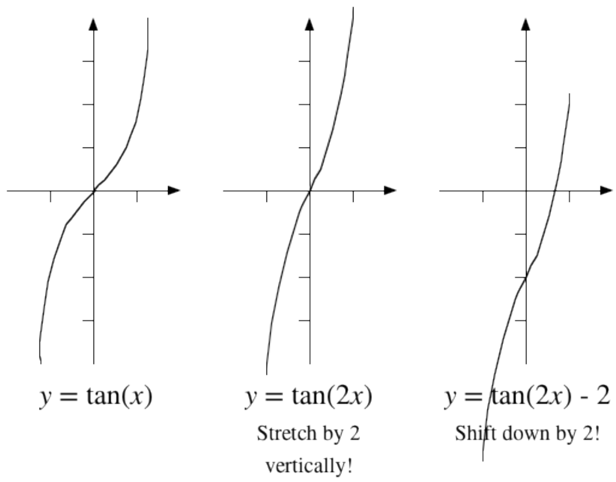
1. Let $f(x) = 2\tan(x) - 2$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Find a formula for $f^{-1}(x)$ and graph both $f(x)$ and $f^{-1}(x)$. *[5]*

SOLUTION. To find a formula for $f^{-1}(x)$, we solve for x in terms of y in the equation $y = 2\tan(x) - 2$,

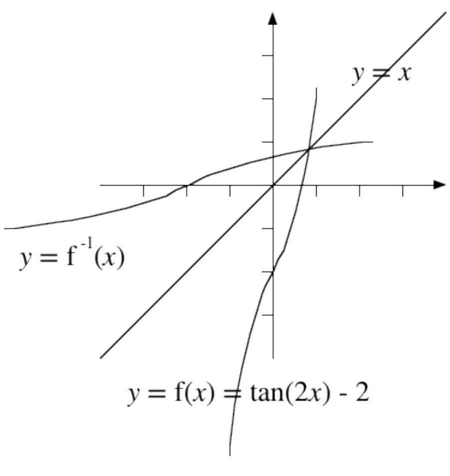
$$y = 2\tan(x) - 2 \iff y + 2 = 2\tan(x) \iff \frac{y+2}{2} = \tan(x)$$
$$\iff \arctan\left(\frac{y+2}{2}\right) = x\,,$$

and then interchange the roles of x and y: $f^{-1}(x) = y = \arctan\left(\frac{x+2}{2}\right)$.

Here is the procedure for generating the graph of $f(x) = 2\tan(x) - 2$ from the graph of $\tan(x)$ (which you should really try to remember). We stick to $-\frac{\pi}{2} < x < \frac{\pi}{2}$, of course:



To get the graph of $f^{-1}(x)$, you can simply reflect the graph of $f(x)$ in the line $y = x$:



Alternatively, you could follow a procedure similar to how the graph of $f(x) = 2\tan(x) - 2$ was obtained above to get the graph of $f^{-1}(x)$ from the graph of $\arctan(x)$, assuming you remember what that looks like. ∎

Quiz #3. Monday, 3 October, 2011. *[10 minutes]*

1. Compute $\lim_{x\to 2} \dfrac{x^2 - x - 2}{\sqrt{x} - \sqrt{2}}$. *[5]* *Hint:* $x^2 - x - 2 = (x-2)(x+1)$.

Solution. If x is positive, which it must be if it is near 2, then $x = \left(\sqrt{x}\right)^2$. It follows that

$$\begin{aligned} x^2 - x - 2 &= (x-2)(x+1) \\ &= \left(\left(\sqrt{x}\right)^2 - \left(\sqrt{2}\right)^2\right)(x+1) \\ &= \left(\sqrt{x} - \sqrt{2}\right)\left(\sqrt{x} + \sqrt{2}\right)(x+1), \end{aligned}$$

so

$$\begin{aligned} \lim_{x\to 2} \frac{x^2 - x - 2}{\sqrt{x} - \sqrt{2}} &= \lim_{x\to 2} \frac{\left(\sqrt{x} - \sqrt{2}\right)\left(\sqrt{x} + \sqrt{2}\right)(x+1)}{\sqrt{x} - \sqrt{2}} \\ &= \lim_{x\to 2} \frac{\left(\sqrt{x} + \sqrt{2}\right)(x+1)}{1} \\ &= \left(\sqrt{2} + \sqrt{2}\right)(2+1) \\ &= 2\sqrt{2} \cdot 3 \\ &= 6\sqrt{2}. \quad \blacksquare \end{aligned}$$

Quiz #4. Tuesday, 11 October, 2011. *[10 minutes]*

1. Explain why $f(x) = \dfrac{\sin(x)}{x}$ is not continuous at $x = 0$ and determine what kind of discontinuity it has there (removable, jump, or vertical asymptote). *[5]*

Solution. $f(x) = \dfrac{\sin(x)}{x}$ cannot be continuous at $x = 0$ because it is not even defined at $x = 0$.

To determine the type of discontinuity it has at $x = 0$ we need to compute and then compare $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$:

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} \frac{\sin(x)}{x} = 1 \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \frac{\sin(x)}{x} = 1 \end{aligned}$$

(As $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, both one-sided limits must exist and also be equal to 1.) Since $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 1$, it follows that $f(x) = \dfrac{\sin(x)}{x}$ has a removable discontinuity at $x = 0$. $\blacksquare$

Quiz #5. Monday, 31 October, 2011. *[10 minutes]*

1. Compute $\frac{dy}{dx}$ if $y = \frac{x^{-1}+x}{e^x}$.

SOLUTION. We throw the Quotient, Sum, and Power Rules, as well as the fact that $\frac{d}{dx}e^x = e^x$, at the problem:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x^{-1}+x}{e^x}\right) \\
&= \frac{\left[\frac{d}{dx}\left(x^{-1}+x\right)\right]\cdot e^x - \left(x^{-1}+x\right)\cdot\left[\frac{d}{dx}e^x\right]}{\left(e^x\right)^2} \\
&= \frac{\left[\frac{d}{dx}x^{-1}+\frac{d}{dx}x\right]\cdot e^x - \left(x^{-1}+x\right)\cdot e^x}{\left(e^x\right)^2} \\
&= \frac{\left[(-1)x^{-2}+1\right]\cdot e^x - \left(x^{-1}+x\right)\cdot e^x}{\left(e^x\right)^2} \\
&= \frac{\left[(-1)x^{-2}+1\right] - \left(x^{-1}+x\right)}{e^x} \\
&= \frac{1-x-x^{-1}-x^{-2}}{e^x} \qquad \blacksquare
\end{aligned}$$

Quiz #6. Monday, 7 November, 2011. *[10 minutes]*

1. Compute $\left.\frac{dy}{dx}\right|_{(x,y)=(0,0)}$ if $x = \sin(x+y)$. *[5]*

SOLUTION. Our main tool will be implicit differentiation. Differentiating both sides of $x = \sin(x+y)$ gives:

$$1 = \frac{d}{dx}x = \frac{d}{dx}\sin(x+y) = \cos(x+y)\cdot\frac{d}{dx}(x+y) = \cos(x+y)\cdot\left(1+\frac{dy}{dx}\right)$$

We solve this for $\frac{dy}{dx}$:

$$\cos(x+y)\cdot\left(1+\frac{dy}{dx}\right) = 1 \Longrightarrow 1+\frac{dy}{dx} = \frac{1}{\cos(x+y)} = \sec x+y \Longrightarrow \frac{dy}{dx} = \sec(x+y)-1$$

Plugging in $(x,y) = (0,0)$ now gives:

$$\left.\frac{dy}{dx}\right|_{(x,y)=(0,0)} = \left(\sec(x+y)-1\right)\Big|_{(x,y)=(0,0)} = \sec(0+0)-1 = \sec(0)-1 = 1-1 = 0$$

Note that $\sec(0) = \frac{1}{\cos(0)} = \frac{1}{1} = 1.$ $\blacksquare$

NOTE: One could also solve for y as a function of x, $y = \arcsin(x) - x$, and then differentiate. This requires knowing, or working out, the derivative of $\arcsin(x)$.

Quiz #7. Monday, 14 November, 2011. *[12 minutes]*

1. Puppies S and E are sniffing a fire hydrant when they are startled by a loud noise, and immediately run off in perpendicular directions. S runs South at 9 m/s and E runs East at 12 m/s. How is the distance between the puppies changing 1 s after they hear the noise?

SOLUTION. Let S and E denote the distance travelled by S and E, respectively. Then $\frac{dS}{dt} = 9\ m/s$ and $\frac{dE}{dt} = 12\ m/s$, so after 1 s we have $S = 9\ m$ and $E = 12\ m$, respectively. At any moment, S and E are the short sides of a right triangle, so the distance between the puppies is $D = \sqrt{S^2 + E^2}$. It follows that

$$\begin{aligned}\frac{dD}{dt} = \frac{d}{dt}\sqrt{S^2+E^2} &= \frac{d}{dt}\left(S^2+E^2\right)^{1/2} = \frac{1}{2}\left(S^2+E^2\right)^{-1/2}\cdot\frac{d}{dt}\left(S^2+E^2\right)\\ &= \frac{1}{2}\left(S^2+E^2\right)^{-1/2}\cdot\left(\frac{d}{dt}S^2+\frac{d}{dt}E^2\right)\\ &= \frac{1}{2}\left(S^2+E^2\right)^{-1/2}\cdot\left(\frac{S^2}{dS}\cdot\frac{dS}{dt}+\frac{E^2}{dE}\cdot\frac{dE}{dt}\right)\\ &= \frac{1}{2}\left(S^2+E^2\right)^{-1/2}\cdot\left(2S\frac{dS}{dt}+2E\frac{dE}{dt}\right) = \frac{S\frac{dS}{dt}+E\frac{dE}{dt}}{\sqrt{S^2+E^2}}\,.\end{aligned}$$

When $t = 1\ s$, we get:

$$\left.\frac{dD}{dt}\right|_{t=1} = \left.\frac{S\frac{dS}{dt}+E\frac{dE}{dt}}{\sqrt{S^2+E^2}}\right|_{t=1} = \frac{9\cdot 9+12\cdot 12}{\sqrt{9^2+12^2}} = \sqrt{9^2+12^2} = \sqrt{81+144} = \sqrt{225} = 15$$

Thus the distance between the puppies is increasing at a rate of 15 m/s 1 s after they hear the noise. ■

Quiz #8. Monday, 21 November, 2011. *[10 minutes]*

1. Find the maxima and minima of $f(x) = 4x^3 - 12x$ on the interval $[0, 2]$. *[5]*

SOLUTION. First, we find the critical points of $f(x)$. Since $f(x)$ is polynomial, it is defined and differentiable everywhere, so we only need to worry about critical points where the derivative is 0.

$$f'(x) = \frac{d}{dx}\left(4x^3 - 12x\right) = 4\cdot 3x^2 - 12\cdot 1 = 12x^2 - 12 = 12(x-1)(x+1)$$

It follows that $f'(x) = 12(x-1)(x+1) = 0$ exactly when $x = 1$ or $x = -1$. Only one of these, $x = 1$, is in $[0, 2]$, so it's the only one we need to consider.

We now check the values of $f(x)$ on the endpoints of the interval and at the critical point in the interval:

x	0	1	2
$f(x)$	0	-8	8

Thus the maximum of $f(x) = 4x^3 - 12x$ on the interval $[0, 2]$ is 8, at the endpoint $x = 2$, and the minimum is -8, at the critical point $x = 1$. ■

Quiz #9. Monday, 28 November, 2011. *[20 minutes]*

1. Find the domain and any (and all!) intercepts, vertical and horizontal asymptotes, local maxima and minima, and points of inflection of $h(x) = \dfrac{x^2-1}{x^2+1}$, and sketch its graph. *[5]*

SOLUTION. We run through the usual checklist in all too much detail, though we won't worry about the range and symmetry of $h(x)$ because they weren't asked for.

i. Domain. $h(x) = \dfrac{x^2-1}{x^2+1}$ is a rational function, so it is defined for all x for which the denominator is not equal to 0. Since $x^2+1 \geq 1 > 0$ for all x, it follows that the domain of $h(x)$ is $\mathbb{R} = (-\infty, +\infty)$. $\square$

ii. Intercepts. $h(0) = \dfrac{0^2-1}{0^2+1} = -1$, so the y-intercept of $h(x)$ is $y = -1$. Since

$$h(x) = \frac{x^2-1}{x^2+1} = 0 \Longleftarrow x^2 - 1 = 0 \Longleftarrow x^2 = 1 \Longleftarrow x = \pm 1\,,$$

$h(x)$ has x-intercepts at $x = \pm 1$. $\square$

iii. Vertical asymptotes. Since $h(x)$ is a rational function, it is continuous everywhere it is defined; since it is defined everywhere, it follows that it has no discontinuities, and hence no vertical asymptotes. $\square$

iv. Horizontal asymptotes. We compute the limits of $h(x)$ as $x \to \pm\infty$:

$$\lim_{x\to-\infty} \frac{x^2-1}{x^2+1} = \lim_{x\to-\infty} \frac{x^2-1}{x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}} = \frac{1-0}{1+0} = 1$$

$$\lim_{x\to+\infty} \frac{x^2-1}{x^2+1} = \lim_{x\to+\infty} \frac{x^2-1}{x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to+\infty} \frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}} = \frac{1-0}{1+0} = 1$$

(Note that $\frac{1}{x^2} \to 0$ as $x \to \pm\infty$.) It follows that $h(x)$ has a horizontal asymptote of $y = 1$ in both directions.

The sharp-eyed may observe that the computation above is somewhat redundant: since $h(x)$ has even symmetry, the limit has to be the same in both directions. In addition, since $x^2 - 1 < x^2 + 1$ for all x, $h(x)$ must approach the asymptote $y = 1$ from below. $\square$

v. Maxima and minima. First, we find $h'(x)$:

$$h'(x) = \frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) = \frac{\frac{d}{dx}\left(x^2-1\right)\cdot\left(x^2+1\right) - \left(x^2-1\right)\cdot\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2}$$

$$= \frac{2x\left(x^2+1\right) - \left(x^2-1\right)2x}{\left(x^2+1\right)^2} = \frac{4x}{\left(x^2+1\right)^2}$$

Second, we find the critical points:

$$h'(x) = \frac{4x}{\left(x^2+1\right)^2} = 0 \Longleftrightarrow 4x = 0 \Longleftrightarrow x = 0$$

(Note that $h'(x)$ is also defined for all x, so we need not consider critical points of the second type, where $h'(x)$ is undefined.) Third, observe that since $\left(x^2+1\right)^2 > 0$ for all x:

$$h'(x) = \frac{4x}{\left(x^2+1\right)^2} \lessgtr 0 \iff 4x \lessgtr 0 \iff x \lessgtr 0$$

Thus, constructing the usual table,

x	$(-\infty, 0)$	0	$(0, \infty)$
$h'(x)$	$-$	0	$+$
$h(x)$	$\downarrow$	min	$\uparrow$

,

we see that $h(x)$ has a local minimum at $x = 0$. Note that $h(0) = -1$. $\square$

vi. Inflection points and concavity. First, we find $h''(x)$:

$$\begin{aligned} h'(x) = \frac{d}{dx}\left(\frac{4x}{\left(x^2+1\right)^2}\right) &= \frac{\frac{d}{dx}(4x)\cdot\left(x^2+1\right)^2 - 4x\cdot\frac{d}{dx}\left(x^2+1\right)^2}{\left(x^2+1\right)^2} \\ &= \frac{4\left(x^2+1\right)^2 - 4x\cdot 2\left(x^2+1\right)2x}{\left(x^2+1\right)^4} \\ &= \frac{4\left(x^2+1\right) - 4x\cdot 2\cdot 2x}{\left(x^2+1\right)^3} = \frac{4-12x^2}{\left(x^2+1\right)^3} \end{aligned}$$

Second, we find the points where $h''(x) = 0$:

$$\begin{aligned} h''(x) = \frac{4-12x^2}{\left(x^2+1\right)^3} = 0 &\iff 4 - 12x^2 = 4\left(1-3x^2\right) = 0 \\ &\iff 3x^2 = 1 \iff x^2 = \frac{1}{3} \iff x = \pm\frac{1}{\sqrt{3}} \end{aligned}$$

(Note that $h''(x)$ is also defined for all x – since $\left(x^2+1\right)^3 \geq 1 > 0$ for all x – so we need not consider potential inflection points where $h''(x)$ is undefined.) Third, observe that since $\left(x^2+1\right)^3 > 0$ for all x:

$$h''(x) = \frac{4-12x^2}{\left(x^2+1\right)^3} \lessgtr 0 \iff 4-12x^2 = 4\left(1-3x^2\right) \lessgtr 0 \iff 3x^2 \gtrless 1 \iff \begin{array}{c} |x| > \frac{1}{\sqrt{3}} \\ |x| < \frac{1}{\sqrt{3}} \end{array}$$

Thus, constructing the usual table,

x	$\left(-\infty, -\frac{1}{\sqrt{3}}\right)$	$-\frac{1}{\sqrt{3}}$	$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\frac{1}{\sqrt{3}}$	$\left(\frac{1}{\sqrt{3}}, \infty\right)$
$h''(x)$	$-$	0	$+$	0	$-$
$h(x)$	$\frown$		$\smile$		$\frown$

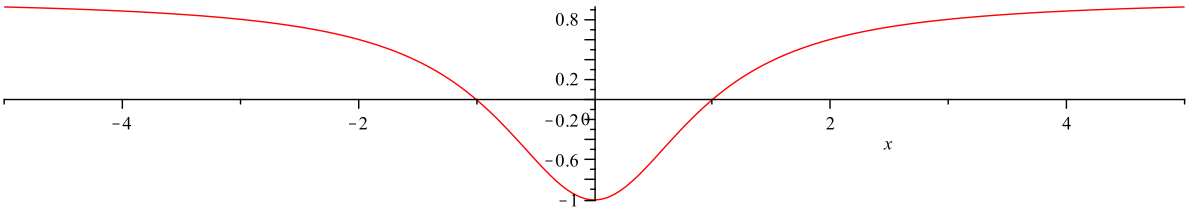
,

we see that $h(x)$ has two inflection points, at $x = \pm\frac{1}{\sqrt{3}}$. $\square$

vi. The graph. We cheat slightly by having `Maple` draw the graph of $h(x)$. The old worksheet-style command

> `plot((x^2-1)/(x^2+1),x=-5..5);`

generates the following graph:



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Quiz #10. Monday, 5 December, 2011. *[12 minutes]*

1. Compute $\displaystyle\int_1^2 x^2\,dx$ using the Right-Hand Rule. *[5]*

SOLUTION. Recall from class that the Right-Hand Rule formula is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n}i\right).$$

We plug the given definite integral into this formula and chug away:

$$\begin{aligned}
\int_1^2 x^2\,dx &= \lim_{n\to\infty} \frac{2-1}{n} \sum_{i=1}^{n} f\left(1 + \frac{2-1}{n}i\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f\left(1 + \frac{1}{n}i\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(1 + \frac{1}{n}i\right)^2 = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[1^2 + 2\cdot 1 \cdot \frac{1}{n}i + \left(\frac{1}{n}i\right)^2\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[1 + \frac{2}{n}i + \frac{1}{n^2}i^2\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\left(\sum_{i=1}^{n} 1\right) + \left(\sum_{i=1}^{n} \frac{2}{n}i\right) + \left(\sum_{i=1}^{n} \frac{1}{n^2}i^2\right)\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[n + \frac{2}{n}\left(\sum_{i=1}^{n} i\right) + \frac{1}{n^2}\left(\sum_{i=1}^{n} i^2\right)\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[n + \frac{2}{n}\cdot\frac{n(n+1)}{2} + \frac{1}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[n + (n+1) + \frac{2n^2+3n+1}{6n}\right] = \lim_{n\to\infty} \frac{1}{n} \left[2n + 1 + \frac{n}{3} + \frac{1}{2} + \frac{1}{6n}\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\frac{7}{3}n + \frac{3}{2} + \frac{1}{6n}\right] = \lim_{n\to\infty} \left[\frac{7}{3} + \frac{3}{2n} + \frac{1}{6n^2}\right] = \frac{7}{3} + 0 + 0 = \frac{7}{3} \quad ■
\end{aligned}$$

Quiz #11. Monday, 9 January, 2012. *[10 minutes]*

1. Compute $\displaystyle\int 2\sin(x)\cos(x)e^{\sin^2(x)}\,dx$. *[5]*

SOLUTION. We will use the Substitution Rule. Let $u = \sin^2(x)$; then

$$\frac{du}{dx} = \frac{d}{dx}\sin^2(x) = 2\sin(x)\cdot\frac{d}{dx}\sin(x) = 2\sin(x)\cos(x)\,,$$

so

$$du = 2\sin(x)\cos(x)\,dx\,,$$

which is conveniently available in the integrand. It follows that

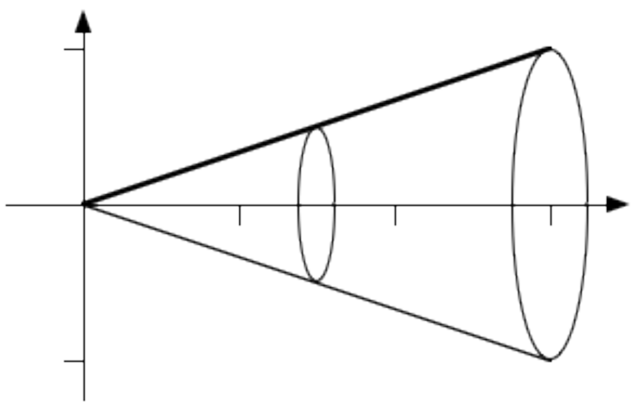
$$\int 2\sin(x)\cos(x)e^{\sin^2(x)}\,dx = \int e^u\,du = e^u + C = e^{\sin^2(x)} + C\,.$$

Note that since we are computing an indefinite integral (*i.e.* a generic antiderivative), we need to include a generic constant. ■

Quiz #12. Monday, 16 January, 2012. *[10 minutes]*

1. Sketch the solid obtained by revolving the region between $y = \frac{1}{3}x$ and $y = 0$ for $0 \le x \le 3$ about the x-axis and find its volume. *[5]*

SOLUTION. Here's a sketch of the solid, a cone with base radius 1 and height 3 placed horizontally instead of vertically:



We will find the volume of the solid by using the disk/washer method. Since we obtained the solid by revolving the region about a horizontal line, namely the x-axis, we will need to integrate with respect to x using the limits 0 to 3 given by the original region. For each x, the cross-section is a washer with outside radius $R = y - 0 = \frac{1}{3}x$ and inside radius $r = 0 - 0 = 0$. Thus the volume of the solid is:

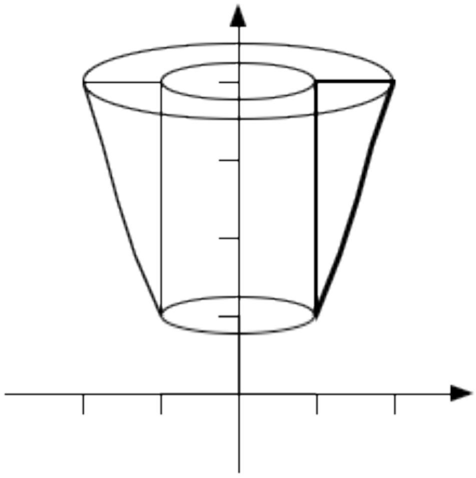
$$\int_0^3 \left(R^2 - r^2\right)\,dx = \int_0^3 \left(\left(\frac{1}{3}x\right)^2 - 0^2\right)\,dx = \int_0^3 \frac{1}{9}x^2\,dx$$

$$= \frac{1}{9}\cdot\frac{x^3}{3}\Big|_0^3 = \frac{1}{27}3^3 - \frac{1}{27}0^3 = \frac{1}{27}27 - 0 = 1 \qquad ■$$

Quiz #13. Monday, 23 January, 2012. *[10 minutes]*

1. Sketch the solid obtained by revolving the region between $y = x^2$ and $y = 4$ for $1 \leq x \leq 2$ about the y-axis and find its volume. *[5]*

SOLUTION. Here's a crude sketch of the solid:



We will find the volume of this solid using both the washer and cylindrical shell methods.

Using washers: Since the axis of revolution is vertical, the washers are horizontal and stacked vertically, which means we will need to integrate with respect to y. Note that the range of possible y values for the region is (since $1^2 = 1$) $1 \leq y \leq 4$. The left edge of the region revolved to make the solid is given by $x = 1$, so the washer for a given y has inside radius $r = x - 0 = 1 - 0 = 1$. Since the right edge of the region is given by $y = x^2$, *i.e.* $x = \sqrt{(y)}$, the outside radius of the washer for a given y is given by $R = x - 0 = \sqrt{y} - 0 = \sqrt{y}$. We plug all this into the integral formula for the volume of the solid:

$$\int_1^4 \pi \left(R^2 - r^2\right) dy = \pi \int_1^4 \left(\left(\sqrt{y}\right)^2 - 1^2\right) dy = \pi \int_1^4 (y-1)\, dy = \pi \left(\frac{y^2}{2} - y\right)\Big|_1^4$$
$$= \pi\left(\frac{4^2}{2} - 4\right) - \pi\left(\frac{1^2}{2} - 1\right) = \pi(8-4) - \pi \cdot \left(-\frac{1}{2}\right) = \frac{9}{2}\pi$$

Using cylindrical shells: Since the axis of revolution is vertical, we need to integrate with respect to the horizontal variable, namely x. The range of possible x values for the region is $1 \leq x \leq 2$ (since $2^2 = 4$). The radius of the washer at x is just $r = x - 0 = x$ and its height is $h = 4 - x^2$. We plug all this into the integral formula for the volume of the solid:

$$\int_1^2 2\pi r h\, dx = 2\pi \int_1^2 x\left(4 - x^2\right) dx = 2\pi \int_1^2 \left(4x - x^3\right) dx = 2\pi \left(4\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_1^2$$
$$= 2\pi\left(2 \cdot 2^2 - \frac{2^4}{4}\right) - 2\pi\left(2 \cdot 1^2 - \frac{1^4}{4}\right) = 2\pi(8-4) - 2\pi\left(2 - \frac{1}{4}\right)$$
$$= 8\pi - \frac{7}{2}\pi = \frac{9}{2}\pi \quad \blacksquare$$

Quiz #14. Monday, 6 February, 2012. *[10 minutes]*

1. Compute $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,dx$. *[5]*

SOLUTION. We will use the trigonometric substitution $x = 2\tan(\theta)$, so $dx = 2\sec^2(\theta)\,d\theta$.

$$\begin{aligned}
\int \frac{1}{\sqrt{4+x^2}}\,dx &= \int \frac{1}{\sqrt{4+(2\tan(\theta))^2}} \cdot 2\sec^2(\theta)\,d\theta = \int \frac{2\sec^2(\theta)}{\sqrt{4+4\tan^2(\theta)}}\,d\theta \\
&= \int \frac{2\sec^2(\theta)}{\sqrt{4\left(1+\tan^2(\theta)\right)}}\,d\theta = \int \frac{2\sec^2(\theta)}{\sqrt{4\sec^2(\theta)}}\,d\theta \\
&= \int \frac{2\sec^2(\theta)}{2\sec(\theta)}\,d\theta = \int \sec(\theta)\,d\theta = \ln\left(\sec(\theta)+\tan(\theta)\right) + C \\
&= \ln\left(\sqrt{1+\tan^2(\theta)}+\tan(\theta)\right) + C = \ln\left(\sqrt{1+\left(\frac{x}{2}\right)^2}+\frac{x}{2}\right) + C \\
&= \ln\left(\sqrt{1+\frac{x^2}{4}}+\frac{x}{2}\right) + C \qquad \blacksquare
\end{aligned}$$

Quiz #15. Monday, 13 February, 2012. *[20 minutes]*

1. Compute $\displaystyle\int \frac{4}{x^3+4x}\,dx$. *[5]*

SOLUTION. We will use partial fractions to take the integral apart. First, we factor the denominator as far as we can: $x^3+4x = x\left(x^2+4\right)$. Note that $x^2+4 \geq 4 > 0$ for all x, so x^2+4 has no roots and so is irreducible. It follows that

$$\int \frac{4}{x^3+4x}\,dx = \int \frac{A}{x}\,dx + \int \frac{Bx+C}{x^2+4}\,dx$$

for some constants A, B, and C we need to determine. Since

$$\frac{4}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x\left(x^2+4\right)} = \frac{A\left(x^2+4\right)+(Bx+C)x}{x\left(x^2+4\right)} = \frac{(A+B)x^2+Cx+4A}{x^3+4x},$$

we must have $A+B=0$, $C=0$, and $4A=4$. C is already nailed down here; from the last of these we get $A=1$, and it now follows from the first that $B=-1$. Hence,

$$\int \frac{4}{x^3+4x}\,dx = \int \frac{1}{x}\,dx + \int \frac{-x+0}{x^2+4}\,dx = \ln(x) - \int \frac{1}{u}\cdot\frac{1}{2}\,du = \ln(x) - \frac{1}{2}\ln(u) + C$$

(where we substituted $u = x^2+4$, so $du = 2\,dx$ and $dx = \frac{1}{2}\,du$)

$$\begin{aligned}
&= \ln(x) - \frac{1}{2}\ln\left(x^2+4\right) + C = \ln(x) - \ln\left(\sqrt{x^2+4}\right) + C \\
&= \ln\left(\frac{x}{\sqrt{x^2+4}}\right) + C \qquad \blacksquare
\end{aligned}$$

Quiz #16. Monday, 27 February, 2012. *[12 minutes]*

1. Find the arc-length of $y = \frac{2}{3}x^{3/2}$ for $0 \le x \le 3$. *[5]*

SOLUTION. $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{2}{3}x^{3/2}\right) = \dfrac{2}{3} \cdot \dfrac{3}{2}x^{1/2} = x^{1/2} = \sqrt{x}$, so

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \left(\sqrt{x}\right)^2}\,dx = \sqrt{1+x}\,dx\,.$$

The arc-length of the curve is therefore given by

$$\begin{aligned}
\int_0^3 ds &= \int_0^3 \sqrt{1+x}\,dx \qquad \text{Substitute } u = 1+x, \text{ so } du = dx \text{ and } \begin{matrix} x & 0 & 3 \\ u & 1 & 4 \end{matrix}. \\
&= \int_1^4 \sqrt{u}\,du = \int_1^4 u^{1/2}\,du = \left.\frac{2}{3}u^{3/2}\right|_1^4 = \frac{2}{3}\left(4^{3/2} - 1^{3/2}\right) \\
&= \frac{2}{3}\left(\left(4^{1/2}\right)^3 - 1\right) = \frac{2}{3}\left(2^3 - 1\right) = \frac{2}{3}(8-1) = \frac{2}{3} \cdot 7 = \frac{14}{3}\,. \qquad \blacksquare
\end{aligned}$$

Quiz #17. Monday, 5 March, 2012. *[10 minutes]*

1. Compute $\displaystyle\lim_{n\to\infty} \frac{\arctan(n)}{n^2}$. *[5]*

SOLUTION. Note that both $\arctan(x)$ and x^2 are defined and continuous on $[1, \infty)$, so

$$\lim_{n\to\infty} \frac{\arctan(n)}{n^2} = \lim_{x\to\infty} \frac{\arctan(x)}{x^2} \begin{matrix} \to \pi/2 \\ \to \infty \end{matrix} = 0$$

since $\arctan(x)$ has a horizontal asymptote of $y = \pi/2$ as $x \to \infty$. $\blacksquare$

Quiz #18. Monday, 12 March, 2012. *[10 minutes]*

1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2+2n-1}$ converges or not. *[5]*

SOLUTION 1. *(Using the (Basic) Comparison Test.)* We will compare the given series to the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. Since

$$\frac{n+1}{n^2+2n-1} = \frac{n+1}{n^2+2n+1-2} = \frac{n+1}{(n+1)^2-2} > \frac{n+1}{(n+1)^2} = \frac{1}{n+1} > 0$$

for all $n \ge 1$, the given series diverges by comparison with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{n=2}^{\infty} \frac{1}{n}$. This last is the harmonic series (less its first term), and so is known to diverge. (One could also use the p-Test to verify the harmonic series diverges.) $\square$

Solution 2. *(Using the Limit Comparison Test.)* Again, we will compare the given series to the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. Since

$$\lim_{n\to\infty} \frac{\frac{n+1}{n^2+2n-1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n+1}{n^2+2n-1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^2+n}{n^2+2n-1} = \lim_{n\to\infty} \frac{n^2+n}{n^2+2n-1} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\frac{n^2}{n^2}+\frac{n}{n^2}}{\frac{n^2}{n^2}+\frac{2n}{n^2}-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}-\frac{1}{n^2}} = \frac{1+0+0}{1+0-0} = 1\,,$$

and $0 < 1 < \infty$, the Limit Comparison Tells us that the given series and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ either both converge or both diverge. Since the harmonic series is known to diverge, this means that $\sum_{n=1}^{\infty} \frac{n+1}{n^2+2n-1}$ must diverge as well. (Again, one could also use the p-Test to verify the harmonic series diverges.) □

Solution 3. *(Using the Integral Test.)* Observe that $a_n = \frac{n+1}{n^2+2n-1} = f(n)$ for the rational function $f(x) = \frac{x+1}{x^2+2x-1}$, which is obviously defined, positive, and continuous on $[1,\infty)$. [We leave it to you to check that it is also decreasing on $[1,\infty)$ – try computing its derivative!] Since the improper integral

$$\int_1^{\infty} \frac{x+1}{x^2+2x-1}\,dx = \lim_{t\to\infty} \int_1^t \frac{x+1}{x^2+2x-1} = \lim_{t\to\infty} \int_2^{t^2+2t-1} \frac{1}{u} \cdot \frac{1}{2}\,du$$

Using the substitution $u = x^2+2x-1$, so $du = (2x+2)\,dx = 2(x+1)\,dx$, with $(x+1)\,dx = \frac{1}{2}\,du$, and $\begin{matrix} x & 1 & t \\ u & 2 & t^2+2t-1 \end{matrix}$.

$$= \lim_{t\to\infty} \frac{1}{2} \int_2^{t^2+2t-1} \frac{1}{u}\,du = \lim_{t\to\infty} \frac{1}{2} \ln(u)\Big|_2^{t^2+2t-1}$$

$$= \lim_{t\to\infty} \left[\ln\left(t^2+2t-1\right) - \ln(2)\right] = \infty\,,$$

($t^2+2t-1 \to \infty$ as $t \to \infty$, and so $\ln\left(t^2+2t-1\right) \to \infty$ as well.)

diverges, it follows by the Integral Test that the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2+2n-1}$ diverges as well. ■