Mathematics 1101Y – Calculus I: functions and calculus of one variable
TRENT UNIVERSITY, 2010–2011

Solutions to the Quizzes

Quiz #1. ~~Friday, 24~~ Monday, 27 September, 2010. *(10 minutes)*

1. Find the location of the tip of the parabola $y = 2x^2 + 2x - 12$, as well as its x- and y-intercepts. *[5]*

SOLUTION. Note that since x^2 has a positive coefficient, this parabola opens upwards.

To find the location of the tip of the parabola, we complete the square in the quadratic expression defining the parabola:

$$\begin{aligned} y &= 2x^2 + 2x - 12 \\ &= 2\left(x^2 + x\right) - 12 \\ &= 2\left[x^2 + 2\frac{1}{2}x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right] - 12 \\ &= 2\left[x^2 + 2\frac{1}{2}x + \left(\frac{1}{2}\right)^2\right] - 2\left(\frac{1}{2}\right)^2 - 12 \\ &= 2\left(x + \frac{1}{2}\right)^2 - \frac{1}{2} - 12 \\ &= 2\left(x + \frac{1}{2}\right)^2 - \frac{25}{2} \end{aligned}$$

It follows that the tip of the parabola occurs when $x + \frac{1}{2} = 0$, *i.e.* when $x = -\frac{1}{2}$, at which point $y = -\frac{25}{2}$. Thus thus the tip of the parabola is at the point $\left(-\frac{1}{2}, -\frac{25}{2}\right)$.

To find the y-intercept of the parabola, we simply plug $x = 0$ into the quadratic expression defining the parabola:

$$y = 2 \cdot 0^2 + 2 \cdot 0 - 12 = 0 + 0 - 12 = 12$$

Thus the y-intercept of the parabola is the point $(0, -12)$.

To find the x-intercept(s) of the parabola, we apply the quadratic formula to find the roots of the quadratic expression defining the parabola: $2x^2 + 2x - 12 = 0$ exactly when

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2 \cdot (-12)}}{2 \cdot 2} = \frac{-2 \pm \sqrt{4 - (-96)}}{4} \\ &= \frac{-2 \pm \sqrt{100}}{4} = \frac{-2 \pm 10}{4} = \frac{-1 \pm 5}{2}, \end{aligned}$$

i.e. exactly when $x = \frac{4}{2} = 2$ or $x = -\frac{6}{2} = -3$. It follows that the parabola has its x-intercepts at $x = 2$ and $x = -3$, *i.e.* at the points $(-3, 0)$ and $(2, 0)$. ■

Quiz #2. Friday, 1 October, 2010. *(6 minutes)*

1. Solve the equation $e^{2x} - 2e^x + 1 = 0$ for x.

Hint: Solve for e^x first ...

Solution. Recall that $e^{2x} = (e^x)^2$, so we can rewrite the given equation as $(e^x)^2 - 2e^x + 1 = 0$. Following the hint, we solve for e^x using the quadratic equation:

$$e^x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{2 \pm \sqrt{4-4}}{2} = \frac{2 \pm 0}{2} = 1$$

Thus $x = \ln(e^x) = \ln(1) = 0$. ■

Quiz #3. Friday, 8 October, 2010. *(10 minutes)*

1. Evaluate the limit $\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1}$, if it exists. *[5]*

Solution. We factor the numerator and simplify:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+2)}{x-1} = \lim_{x \to 1} (x+2) = 1 + 2 = 3$$

In case of problems factoring this by sight, one could always apply the quadratic formula. The roots of $x^2 + x - 2$ are:

$$\frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-2)}}{2 \cdot 1} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2} = +1 \text{ or } -2$$

It follows that $x^2 + x - 2 = (x-1)(x-(-2)) = (x-1)(x+2)$. ■

Quiz #4. Friday, 15 October, 2010. *(10 minutes)*

1. Use the limit definition of the derivative to compute $f'(2)$ if $f(x) = x^2 + 3x + 1$. *[5]*

Solution. Here goes!

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0} \frac{\left[(2+h)^2 + 3(2+h) + 1\right] - \left[2^2 + 3 \cdot 2 + 1\right]}{h} \\ &= \lim_{h \to 0} \frac{\left[4 + 4h + h^2 + 6 + 3h + 1\right] - \left[4 + 6 + 1\right]}{h} \\ &= \lim_{h \to 0} \frac{h^2 + 7h}{h} \\ &= \lim_{h \to 0} (h + 7) = 0 + 7 = 7 \qquad \blacksquare \end{aligned}$$

Quiz #5. ~~Friday, 22 October~~ Monday, 1 November, 2010. *(10 minutes)*

1. Find $f'(x)$ if $f(x) = \frac{x^2+2x}{x^2+2x+1}$. Simplify your answer as much as you reasonably can. *[5]*

SOLUTION. The Quotient Rule followed by algebra:

$$\begin{aligned}
f'(x) &= \frac{\frac{d}{dx}\left(x^2+2x\right)\cdot\left(x^2+2x+1\right) - \left(x^2+2x\right)\cdot\frac{d}{dx}\left(x^2+2x+1\right)}{\left(x^2+2x+1\right)^2} \\
&= \frac{(2x+2)\left(x^2+2x+1\right) - \left(x^2+2x\right)(2x+2)}{\left(x^2+2x+1\right)^2} \\
&= \frac{2(x+1)(x+1)^2 - \left(x^2+2x\right)2(x+1)}{\left((x+1)^2\right)^2} \\
&= \frac{2(x+1)^3 - \left(x^2+2x\right)2(x+1)}{(x+1)^4} \\
&= \frac{2(x+1)^2 - 2\left(x^2+2x\right)}{(x+1)^3} \\
&= \frac{2\left(x^2+2x+1\right) - 2\left(x^2+2x\right)}{(x+1)^3} \\
&= \frac{2}{(x+1)^3} \qquad \blacksquare
\end{aligned}$$

Quiz #6. Friday, 5 November, 2010. *(10 minutes)*

1. Find $\frac{dy}{dx}$ if $y = \sqrt{x + \arctan(x)}$. *[5]*

SOLUTION. This is a job for the Chain Rule. Note first that, using the Power Rule, $\frac{d}{dt}\sqrt{t} = \frac{d}{dt}t^{1/2} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}}$. Letting $t = x + \arctan(x)$ and applying the Chain Rule gives:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dt}\cdot\frac{dt}{dx} = \left(\frac{d}{dt}\sqrt{t}\right)\cdot\frac{dt}{dx} = \frac{1}{2\sqrt{t}}\cdot\frac{dt}{dx} \\
&= \frac{1}{2\sqrt{x+\arctan(x)}}\cdot\frac{d}{dx}\left(x+\arctan(x)\right) \\
&= \frac{1}{2\sqrt{x+\arctan(x)}}\cdot\left(\frac{dx}{dx}+\frac{d}{dx}\arctan(x)\right) \\
&= \frac{1}{2\sqrt{x+\arctan(x)}}\cdot\left(1+\frac{1}{1+x^2}\right)
\end{aligned}$$

There's not much one can to do to meaningfully simplify this. A little algebra could give you something like $\frac{dy}{dx} = \frac{2+x^2}{2(1+x^2)\sqrt{x+\arctan(x)}}$, but it's not clear that's an improvement. $\blacksquare$

Quiz #7. Friday, 12 November, 2010. *(10 minutes)*

1. Find the maximum and minimum of $f(x) = \dfrac{x}{1+x^2}$ on the interval $[-2, 2]$. *[5]*

Solution. We compute $f'(x)$ using the Quotient Rule:

$$f'(x) = \frac{\left(\frac{d}{dx}x\right)\left(1+x^2\right) - x\frac{d}{dx}\left(1+x^2\right)}{\left(1+x^2\right)^2} = \frac{1\left(1+x^2\right) - x \cdot 2x}{\left(1+x^2\right)^2} = \frac{1-x^2}{\left(1+x^2\right)^2}$$

Note that the denominator of $f'(x)$ is never 0 because $1 + x^2 \geq 1$ for all x, so $f'(x)$ is defined for all x in the interval $[-2, 2]$. $f'(x) = \dfrac{1-x^2}{\left(1+x^2\right)^2} = 0$ exactly when $1 - x^2 = 0$. It follows that the critical points of $f(x)$ are $x = \pm 1$, both of which in the interval $[-2, 2]$.

We compare the values of $f(x)$ at the critical points and the endpoints of the interval:

x	$f(x) = \dfrac{x}{1+x^2}$
-2	$\dfrac{-2}{1+(-2)^2} = -\dfrac{2}{5}$
-1	$\dfrac{-1}{1+(-1)^2} = -\dfrac{1}{2}$
1	$\dfrac{1}{1+1^2} = \dfrac{1}{2}$
2	$\dfrac{2}{1+2^2} = \dfrac{2}{5}$

Since $-\dfrac{1}{2} < -\dfrac{2}{5} < \dfrac{2}{5} < \dfrac{1}{2}$, it follows that the maximum of $f(x) = \dfrac{x}{1+x^2}$ on the interval $[-2, 2]$ is $f(1) = \dfrac{1}{2}$ and the minimum is $f(-1) = -\dfrac{1}{2}$. ■

Quiz #8. Friday, 26 November, 2010. *(10 minutes)*

1. Find an antiderivative of $f(x) = 4x^3 - 3\cos(x) + \dfrac{1}{x}$. *[5]*

Solution. This is mainly an exercise in memorizing basic rules about antiderivatives and the antiderivatives of standard functions. Using the indefinite integral notation for antiderivatives we get:

$$\begin{aligned}\int \left(4x^3 - 3\cos(x) + \frac{1}{x}\right) dx &= 4\int x^3\,dx - 3\int \cos(x)\,dx + \int \frac{1}{x}\,dx \\ &= 4 \cdot \frac{x^{3+1}}{3+1} - 3\sin(x) + \ln(x) + C \\ &= x^4 - 3\sin(x) + \ln(x) + C\end{aligned}$$

Since we just asked for *an* antiderivative, any value of C – including 0 – is fine here. ■

Quiz #9. Friday, 3 December, 2010. *(10 minutes)*

1. Compute the definite integral $\int_0^1 (2x+1)\, dx$ using the Right-hand Rule. *[5]*

Hint: You may assume that $\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

SOLUTION. Recall that the Right-hand Rule formula is:

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{b-a}{n} f\left(a + k\frac{b-a}{n}\right)$$

We plug $a = 0$, $b = 1$, and $f(x) = 2x + 1$ into this formula and grind away:

$$\begin{aligned}
\int_0^1 (2x+1)\, dx &= \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1-0}{n} f\left(0 + k\frac{1-0}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{n} f\left(\frac{k}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} f\left(\frac{k}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} \left(2\frac{k}{n} + 1\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\left(\sum_{k=1}^{\infty} \frac{2k}{n}\right) + \left(\sum_{k=1}^{\infty} 1\right)\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\left(\frac{2}{n} \sum_{k=1}^{\infty} k\right) + \left(\sum_{k=1}^{\infty} 1\right)\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\frac{2}{n} \cdot \frac{n(n+1)}{2} + n\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[(n+1) + n\right] \\
&= \lim_{n\to\infty} \frac{1}{n} \left[2n+1\right] = \lim_{n\to\infty} \left[\frac{2n}{n} + \frac{1}{n}\right] \\
&= \lim_{n\to\infty} \left[2 + \frac{1}{n}\right] = 2 + 0 = 2 \qquad \blacksquare
\end{aligned}$$

Quiz #10. Friday, 10 December, 2010. *(10 minutes)*

1. Find the area between the graphs of $f(x) = \sin(x)$ and $g(x) = \dfrac{2x}{\pi}$ for $0 \le x \le \dfrac{\pi}{2}$. *[5]*

Solution. Note that $f(0) = 0 = g(0)$ and $f\left(\frac{\pi}{2}\right) = g\left(\frac{\pi}{2}\right) = 1$. Between these points $f(x) \ge g(x)$ – sketch the graphs to convince yourself, if necessary – so the area between them is:

$$\begin{aligned}
\int_0^{\pi/2} (f(x) - g(x))\, dx &= \int_0^{\pi/2} \left(\sin(x) - \frac{2x}{\pi}\right) dx \\
&= \left(-\cos(x) - \frac{2}{\pi} \cdot \frac{x^2}{2}\right)\Bigg|_0^{\pi/2} \\
&= \left(-\cos(\pi/2) - \frac{(\pi/2)^2}{\pi}\right) - \left(-\cos(0) - \frac{0^2}{\pi}\right) \\
&= \left(-0 - \frac{\pi}{4}\right) - (-1 - 0) \\
&= -\frac{\pi}{4} + 1 = 1 - \frac{\pi}{4}
\end{aligned}$$

Quick sanity check: $\pi < 4$, so $\frac{\pi}{4} < 1$, so $1 - \frac{\pi}{4}$ is positive, as an area should be. ■

Quiz #11. Friday, 14 January, 2011. *(10 minutes)*

1. Compute $\displaystyle\int_0^{\pi/2} \cos^3(x)\, dx$. *[5]*

Solution. This can be done pretty quickly using the appropriate reduction formula or integration by parts, but it's also easy to do by a combination of the trig identity $\cos^2(x) = 1 - \sin^2(x)$ and substitution:

$$\begin{aligned}
\int_0^{\pi/2} \cos^3(x)\, dx &= \int_0^{\pi/2} \cos^2(x)\cos(x)\, dx \\
&= \int_0^{\pi/2} \left(1 - \sin^2(x)\right)\cos(x)\, dx
\end{aligned}$$

Substitute $u = \sin(x)$, so $du = \cos(x)\, dx$, and change the limits: $\begin{matrix} x & 0 & \pi/2 \\ u & 0 & 1 \end{matrix}$.

$$\begin{aligned}
&= \int_0^1 \left(1 - u^2\right) du \\
&= \left(u - \frac{u^3}{3}\right)\Bigg|_0^1 \\
&= \left(1 - \frac{1^3}{3}\right) - \left(0 - \frac{0^3}{3}\right) \\
&= \frac{2}{3} - 0 = \frac{2}{3} \qquad ■
\end{aligned}$$

Quiz #12. Friday, 21 January, 2011. *(10 minutes)*

1. Compute $\displaystyle\int \tan^3(x)\sec(x)\,dx$. *[5]*

SOLUTION. There are other ways to pull this off, but the following use of the identity $\tan^2(x) = \sec^2(x) - 1$ and substitution is pretty quick:

$$\begin{aligned}
\int \tan^3(x)\sec(x)\,dx &= \int \tan^2(x)\tan(x)\sec(x)\,dx \\
&= \int \left(\sec^2(x) - 1\right)\sec(x)\tan(x)\,dx \\
&\qquad \text{Substitute } u = \sec(x)\text{, so } du = \sec(x)\tan(x)\,dx. \\
&= \int \left(u^2 - 1\right)\,dx \\
&= \frac{1}{3}u^3 - u + C \\
&= \frac{1}{3}\sec^3(x) - \sec(x) + C \qquad \blacksquare
\end{aligned}$$

Quiz #13. Friday, 28 January, 2011. *(10 minutes)*

1. Compute $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,dx$. *[5]*

SOLUTION. We'll use the trigonometric substitution $x = 2\tan(\theta)$, so $dx = 2\sec^2(\theta)\,d\theta$, and also $\tan(\theta) = \frac{x}{2}$ and $\sec(\theta) = \sqrt{\sec^2(\theta)} = \sqrt{1+\tan^2(\theta)} = \sqrt{1+\frac{x^2}{4}}$. (We'll need these last when substituting back.)

$$\begin{aligned}
\int \frac{1}{\sqrt{4+x^2}}\,dx &= \int \frac{1}{\sqrt{4+(2\tan(\theta))^2}}\,2\sec^2(\theta)\,d\theta \\
&= \int \frac{2\sec^2(\theta)}{\sqrt{4+4\tan^2(\theta)}}\,d\theta = \int \frac{2\sec^2(\theta)}{\sqrt{4\left(1+\tan^2(\theta)\right)}}\,d\theta \\
&= \frac{2}{\sqrt{4}}\int \frac{\sec^2(\theta)}{\sqrt{1+\tan^2(\theta)}}\,d\theta = \frac{2}{2}\int \frac{\sec^2(\theta)}{\sqrt{\sec^2(\theta)}}\,d\theta \\
&= \int \frac{\sec^2(\theta)}{\sec(\theta)}\,d\theta = \int \sec(\theta)\,d\theta \\
&= \ln\left(\sec(\theta) + \tan(\theta)\right) + C \\
&= \ln\left(\sqrt{1+\frac{x^2}{4}} + \frac{x}{2}\right) + C \qquad \blacksquare
\end{aligned}$$

Quiz #14. Friday, 4 February, 2011. *(15 minutes)*

1. Compute $\displaystyle\int \frac{4x^2+3x}{(x+2)\left(x^2+1\right)}\,dx$. *[5]*

SOLUTION. This is a job for partial fractions. Note that the denominator of the integrand, $(x+2)\left(x^2+1\right)$, come pre-factored into linear factor and irreducible quadratic factors. (x^2+1 doesn't factor any further because it has no roots, since $x^2+1 \geq 1 > 0$ for all x.) It follows that

$$\frac{4x^2+3x}{(x+2)\left(x^2+1\right)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

for some constants A, B, and C. To determine these constants we put the right-hand side of the above equation over the common denominator

$$\begin{aligned}\frac{A}{x+2} + \frac{Bx+C}{x^2+1} &= \frac{A\left(x^2+1\right)+(Bx+C)(x+2)}{(x+2)\left(x^2+1\right)}\\ &= \frac{Ax^2+A+Bx^2+2Bx+Cx+2C}{(x+2)\left(x^2+1\right)}\\ &= \frac{(A+B)x^2+(2B+C)x+(A+2C)}{(x+2)\left(x^2+1\right)},\end{aligned}$$

and then equate numerators

$$4x^2+3x = (A+B)x^2+(2B+C)x+(A+2C)$$

to obtain a set of three linear equations in A, B, and C:

$$\begin{array}{ccccccc} A & + & B & & & = & 4\\ & & 2B & + & C & = & 3\\ A & & & + & 2C & = & 0 \end{array}$$

These equations can be solved in a variety of ways; we will do so using substitution. Solving the third equation for A gives $A = -2C$. Substituting this into the first equation gives $B - 2C = 4$; solving this for B now gives $B = 4 + 2C$. Substituting the last into the second equation now gives $2(4+2C)+C = 3$, *i.e.* $8 + 5C = 3$, so $5C = 3 - 8 = -5$, so $C = -5/5 = -1$. It follows that $B = 4+2C = 4+2(-1) = 2$ and $A = -2C = -2(-1) = 2$.

Thus

$$\begin{aligned}\int \frac{4x^2+3x}{(x+2)\left(x^2+1\right)}\,dx &= \int \left(\frac{2}{x+2}+\frac{2x-1}{x^2+1}\right)dx\\ &= \int \frac{2}{x+2}\,dx + \int \frac{2x-1}{x^2+1}\,dx\\ &= \int \frac{2}{x+2}\,dx + \int \frac{2x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx\end{aligned}$$

We handle the three parts separately. In the first, we use the substitution $u = x + 2$, so $du = dx$. In the second, we use the substitution $w = x^2 + 1$, so $dw = 2x\,dx$. In the third, we recollect that $\dfrac{1}{x^2+1}$ is the derivative of $\arctan(x)$. Now

$$\begin{aligned}
\int \frac{4x^2+3x}{(x+2)\left(x^2+1\right)}\,dx &= \int \frac{2}{x+2}\,dx + \int \frac{2x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx \\
&= 2\int \frac{1}{u}\,du + \int \frac{1}{w}\,dw + \arctan(x) \\
&= 2\ln(u) + \ln(w) + \arctan(x) + C \\
&\qquad \text{Since the last integral sign has disappeared, the generic} \\
&\qquad \text{constant must now show up. Substituting back gives:} \\
&= 2\ln(x+2) + \ln\left(x^2+1\right) + \arctan(x) + C
\end{aligned}$$

Whew! ■

~~**Quiz #15.** Friday, 18 February, 2011. *(15 minutes)*~~

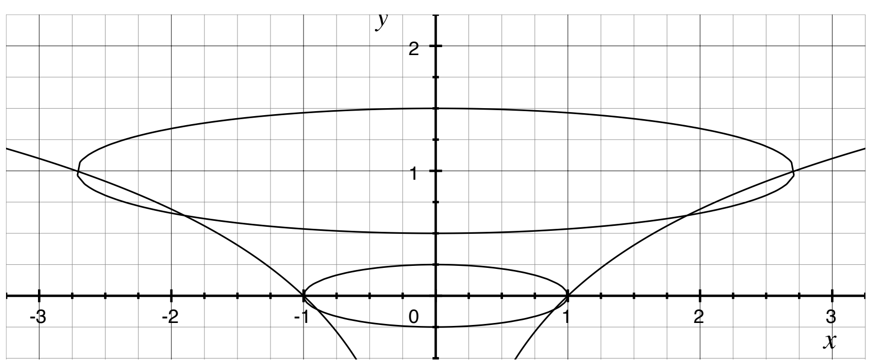
1. Sketch the surface obtained by revolving the curve $y = \ln(x)$, $1 \le x \le e$, about the y-axis, and find its area. *[5]*

Hint: You may find it convenient to just use the fact that

$$\int \sec^3(\theta)\,d\theta = \frac{1}{2}\tan(\theta)\sec(\theta) + \frac{1}{2}\ln\left(\tan(\theta)+\sec(\theta)\right) + C\,,$$

instead of having to work it out from scratch.

Solution. Here's a sketch of the surface, albeit I cheated a little by starting with a graph of $y = \ln(x)$ drawn by a computer.



Note that we only want the part that come from revolving the part of $y = \ln(x)$ for $1 \le x \le e$.

To find the area of this surface, we use the usual formula for the area of a surface of revolution obtained by revolving a curve about the y-axis:

$$\begin{aligned}
\int_1^e 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx &= 2\pi\int_1^e x\sqrt{1+\left(\frac{d}{dx}\ln(x)\right)^2}\,dx = 2\pi\int_1^e x\sqrt{1+\left(\frac{1}{x}\right)^2}\,dx \\
&= 2\pi\int_1^e x\sqrt{1+\frac{1}{x^2}}\,dx = 2\pi\int_1^e \sqrt{x^2\left(1+\frac{1}{x^2}\right)}\,dx \\
&= 2\pi\int_1^e \sqrt{x^2+1}\,dx
\end{aligned}$$

Substitute $x=\tan(\theta)$, so $dx=\sec^2(\theta)\,d\theta$,

keep the old limits and substitute back.

$$\begin{aligned}
&= 2\pi\int_{x=1}^{x=e} \sqrt{\tan^2(\theta)+1}\,\sec^2(\theta)\,d\theta \\
&= 2\pi\int_{x=1}^{x=e} \sqrt{\sec^2(\theta)}\,\sec^2(\theta)\,d\theta = 2\pi\int_{x=1}^{x=e}\sec^3(\theta)\,d\theta
\end{aligned}$$

Use the hint! Note that $\sec(\theta)=\sqrt{\tan^2(\theta)+1}=\sqrt{x^2+1}$.

$$\begin{aligned}
&= 2\pi\left[\frac{1}{2}\tan(\theta)\sec(\theta)+\frac{1}{2}\ln\left(\tan(\theta)+\sec(\theta)\right)\right]\Bigg|_{x=1}^{x=e} \\
&= \pi\left[x\sqrt{x^2+1}+\ln\left(x+\sqrt{x^2+1}\right)\right]\Bigg|_{x=1}^{x=e} \\
&= \pi\left[e\sqrt{e^2+1}+\ln\left(e+\sqrt{e^2+1}\right)-\sqrt{2}-\ln\left(1+\sqrt{2}\right)\right]
\end{aligned}$$

This doesn't seem to simplify nicely . . . ■

Quiz #15. Some time or other, 2011. *(15 minutes)*

1. Find the area of the surface obtained by revolving the curve $y=\sqrt{1-x^2}$, where $0\le x\le 1$, about the y-axis. *[5]*

SOLUTION. In this case

$$\frac{dy}{dx}=\frac{d}{dx}\sqrt{1-x^2}=\frac{1}{2\sqrt{1-x^2}}\cdot\frac{d}{dx}\left(1-x^2\right)=\frac{1}{2\sqrt{1-x^2}}\cdot(-2x)=\frac{-x}{\sqrt{1-x^2}}\,.$$

We plug this into the surface area formula $\int_a^b 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$, giving:

$$\begin{aligned}
\text{Area} &= \int_0^1 2\pi x\sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx = \int_0^1 2\pi x\sqrt{1+\frac{x^2}{1-x^2}}\,dx \\
&= \int_0^1 2\pi x\sqrt{\frac{1-x^2}{1-x^2}+\frac{x^2}{1-x^2}}\,dx = \int_0^1 2\pi x\sqrt{\frac{1-x^2+x^2}{1-x^2}}\,dx \\
&= \int_0^1 2\pi x\sqrt{\frac{1}{1-x^2}}\,dx = \int_0^1 2\pi x\frac{1}{\sqrt{1-x^2}}\,dx
\end{aligned}$$

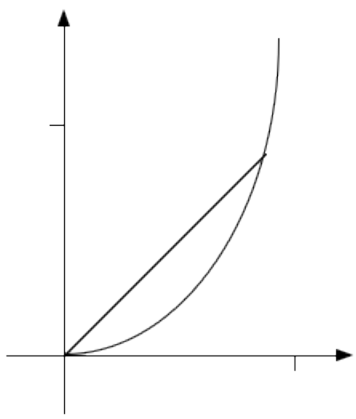
We will compute the last integral using the substitution $u = 1 - x^2$, so $du = -2x\,dx$ and $(-1)\,du = 2x\,dx$, changing the limits as we go along: $\begin{matrix} x & 0 & 1 \\ u & 1 & 0 \end{matrix}$. Then:

$$\text{Area} = \int_0^1 2\pi x \frac{1}{\sqrt{1-x^2}}\,dx = \int_1^0 \pi \frac{1}{\sqrt{u}}(-1)\,du = \pi \int_0^1 u^{1/2}\,du$$
$$= \pi \frac{u^{3/2}}{3/2}\Big|_0^1 = \pi \frac{2}{3} u^{3/2}\Big|_0^1 = \pi\frac{2}{3}1^{3/2} - \pi\frac{2}{3}0^{3/2} = \frac{2}{3}\pi \qquad \blacksquare$$

Quiz #16. Some time or other, 2011. *(12 minutes)*

1. Sketch the region bounded by $r = \tan(\theta)$, $\theta = 0$, and $\theta = \frac{\pi}{4}$ in polar coordinates and find its area. *[5]*

Solution. Note that when $\theta = 0$, $r = \tan(0) = 0$, and when $\theta = \frac{\pi}{4}$, $r = \tan(\pi/4) = \frac{\sin(\pi/4)}{\cos(\pi/4)} = \frac{1/\sqrt{2}}{1/\sqrt{2}} = 1$. In between, $\sin(\theta)$ is increasing and $\cos(\theta)$ is decreasing as θ is increasing, so $r = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ is increasing. The region therefore looks something like this:



To find its area, we need to plug $r = \tan(\theta)$ into the polar area formula for $0 \le \theta \le \frac{\pi}{4}$ and integrate away:

$$\text{Area} = \int_\alpha^\beta \frac{1}{2} r^2\,d\theta = \int_0^{\pi/4} \frac{1}{2}\tan^2(\theta)\,d\theta = \frac{1}{2}\int_0^{\pi/4} \left(\sec^2(\theta) - 1\right)\,d\theta = \frac{1}{2}\left(\tan(\theta) - \theta\right)\Big|_0^{\pi/4}$$
$$= \frac{1}{2}\left(\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\right) - \frac{1}{2}\left(\tan(0) - 0\right) = \frac{1}{2}\left(1 - \frac{\pi}{4}\right) - \frac{1}{2}0 = \frac{1}{2} - \frac{\pi}{8} \qquad \blacksquare$$

Quiz #17. Friday, 11 March, 2011. *(12 minutes)*

1. Find the arc-length of the parametric curve $x = \sec(t)$, $y = \ln(\sec(t) + \tan(t))$, where $0 \le t \le \frac{\pi}{4}$.

SOLUTION. We're going to need to know $\frac{dx}{dt}$ and $\frac{dy}{dt}$, so we'll compute them first:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{d}{dt}\sec(t) = \sec(t)\tan(t) \\
\frac{dy}{dt} &= \frac{d}{dt}\ln(\sec(t) + \tan(t)) \\
&= \frac{1}{\sec(t) + \tan(t)} \cdot \frac{d}{dt}(\sec(t) + \tan(t)) \\
&= \frac{1}{\sec(t) + \tan(t)} \cdot \left(\sec(t)\tan(t) + \sec^2(t)\right) \\
&= \frac{1}{\sec(t) + \tan(t)} \cdot \sec(t)(\tan(t) + \sec(t)) = \sec(t)
\end{aligned}$$

We can now compute the arc-length of the given curve:

$$\begin{aligned}
\text{arc-length} &= \int_C ds = \int_0^{\pi/4} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^{\pi/4} \sqrt{(\sec(t)\tan(t))^2 + (\sec(t))^2}\, dt \\
&= \int_0^{\pi/4} \sqrt{\sec^2(t)\left[\tan^2(t) + 1\right]}\, dt = \int_0^{\pi/4} \sqrt{\sec^2(t)\sec^2(t)}\, dt \\
&= \int_0^{\pi/4} \sqrt{\left(\sec^2(t)\right)^2}\, dt = \int_0^{\pi/4} \sec^2(t)\, dt = \tan(t)\Big|_0^{\pi/4} \\
&= \tan\left(\frac{\pi}{4}\right) - \tan(0) = 1 - 0 = 1 \qquad \blacksquare
\end{aligned}$$

Quiz #18. Friday, 18 March, 2011. *(10 minutes)*

1. Compute $\displaystyle\lim_{n\to\infty} \frac{n^2}{e^n}$. *[5]*

Solution. Observe that $f(x) = \dfrac{x^2}{e^x}$ is defined and differentiable (and hence continuous) on $[0, \infty)$, and such that $f(n) = \dfrac{n^2}{e^n}$. It follows that:

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2}{e^n} &= \lim_{x\to\infty} \frac{x^2}{e^x} && \text{Since } x^2 \to \infty \text{ and } e^x \to \infty \text{ as } x \to \infty, \\
& && \text{we apply L'Hôpital's Rule.} \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}x^2}{\frac{d}{dx}e^x} \\
&= \lim_{x\to\infty} \frac{2x}{e^x} && \text{Since } 2x \to \infty \text{ and } e^x \to \infty \text{ as } x \to \infty, \\
& && \text{we apply L'Hôpital's Rule again.} \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}2x}{\frac{d}{dx}e^x} \\
&= \lim_{x\to\infty} \frac{2}{e^x} \\
&= 0 && \ldots \text{ because 2 is constant and } e^x \to \infty \text{ as } x \to \infty. \ \blacksquare
\end{aligned}$$

Quiz #19. Friday, 25 March, 2011. *(10 minutes)*

1. Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2 + 2^n}$ converges or diverges. *[5]*

Solution. Note that $0 < \dfrac{1}{n^2 + 2^n} \le \dfrac{1}{2^n} = \left(\dfrac{1}{2}\right)^n$ for all $n \ge 0$, and that the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges, because it is a geometric series with common ratio $\frac{1}{2} < 1$. It follows that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2 + 2^n}$ converges by the Comparison Test. $\blacksquare$

Note: One could also compare the given series to the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2}$, which converges by the p-Test.

Quiz #20. Friday, 1 April, 2011. *(15 minutes)*

1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$ converges absolutely, converges conditionally, or diverges.

Solution. This series converges conditionally.

First, we check if the given series converges absolutely. The corresponding series of positive terms is $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^n \ln(n)}{n} \right| = \sum_{n=2}^{\infty} \frac{\ln(n)}{n}$. Since $f(x) = \dfrac{\ln(x)}{x}$ is defined, continuous, and positive for $x \geq 2$, we can use the Integral Test. Since the improper integral

$$\int_2^{\infty} \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty} \int_2^t \frac{\ln(x)}{x}\,dx \qquad \text{(Substitute } u = \ln(x)\text{, so } du = \tfrac{1}{x}\,dx\text{, and change limits accordingly.)}$$

$$= \lim_{t\to\infty} \int_{\ln(2)}^{\ln(t)} u\,du = \lim_{t\to\infty} u^2 \Big|_{\ln(2)}^{\ln(t)} = \lim_{t\to\infty} \left[(\ln(t))^2 - (\ln(2))^2 \right]$$

$$= \infty \qquad \text{(Since } \ln(t) \to \infty \text{ as } t \to \infty\text{.)}$$

does not converge, it follows by the Integral Test that neither does $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$. (One could also do this part very quickly by comparison with the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$.) Thus the given series does not converge absolutely.

Second, to check that the given series converges, we will apply the Alternating Series Test.

- The series is indeed alternating: once $n > 1$, $\dfrac{\ln(n)}{n}$ is always positive, so the $(-1)^n$ forces successive terms to switch sign.
- $\lim_{n\to\infty} \left| \frac{(-1)^n \ln(n)}{n} \right| = \lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$, as required. (Note the use of l'Hôpital's Rule at the key step.)
- Successive terms decrease in absolute value once $n > 1$ since the function $f(x) = \frac{\ln(x)}{x}$ is decreasing for $x > e$ because (Quotient Rule!) $f'(x) = \frac{\frac{1}{x}x - \ln(x)1}{x^2} = \frac{1-\ln(x)}{x^2} < 0$ as soon as $\ln(x) > 1$.

It follows that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$ converges by the Alternating Series Test. Since it does converge, but not absolutely, the series converges conditionally. ■