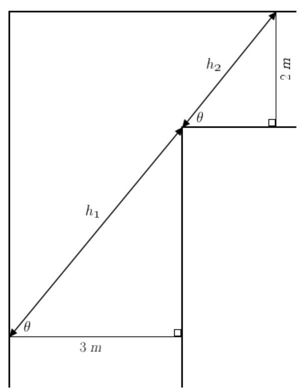
Mathematics 1101Y – Calculus I: functions and calculus of one variable
Trent University, 2010–2011

Solutions to Assignment #6
Pipe dreams?

1. A rigid pipe is carried down a corridor 3 m wide. At the end of this corridor there is a right-angled turn into another corridor that is 2 m wide. What is the length of the longest pipe that can be carried horizontally around the corner? *[10]*

Solution. The length of the longest pipe that can be carried horizontally around the corner without getting stuck is the length of the *shortest* line segment that can be placed so that its ends touch the corridor walls and some point on the line segment touches the corner. There are several ways to set this up as a max/min problem; the one we'll use here will take advantage of the fact that a line segment placed as above can be thought of as being made up of the hypotenuses of two right triangles as in the diagram below:



We will use the angle θ that the line segment makes with the interior wall of the narrower corridor as our underlying variable.

First, observe that θ is also the angle that the line segment makes with a perpendicular to the wall of the wider corridor. This allows us to express both h_1 and h_2 in terms of θ by using the properties of trig functions and right triangles:

$$\cos(\theta) = \tfrac{3}{h_1} \quad \Longrightarrow \quad h_1 = \frac{3}{\cos(\theta)}$$
$$\sin(\theta) = \tfrac{2}{h_2} \quad \Longrightarrow \quad h_2 = \frac{2}{\sin(\theta)}$$

Thus the length of the line segment in terms of θ is:

$$L(\theta) = h_1 + h_2 = \frac{3}{\cos(\theta)} + \frac{2}{\sin(\theta)}$$

Second, note that $0 < \theta < \frac{\pi}{2}$. θ cannot actually achieve either extreme since the line segment would be infinitely long and not actually touching both of the outside corridor walls when $\theta = 0$ or $\theta = \frac{\pi}{2}$. Since $\cos\left(\frac{\pi}{2}\right) = \sin(0) = 0$, it is clear that $L(\theta)$ will tend to infinity as θ approaches either 0 or $\frac{\pi}{2}$.

We now find the minimum value of $L(\theta)$ for $0 < \theta < \frac{\pi}{2}$ using our usual max/min technology:

$$\begin{aligned}
L'(\theta) &= \frac{d}{d\theta}\left(\frac{3}{\cos(\theta)} + \frac{2}{\sin(\theta)}\right) = -\frac{3}{\cos^2(\theta)} \cdot \frac{d}{d\theta}\cos(\theta) - \frac{2}{\sin^2(\theta)} \cdot \frac{d}{d\theta}\sin(\theta) \\
&= -\frac{3}{\cos^2(\theta)} \cdot (-\sin(\theta)) - \frac{2}{\sin^2(\theta)} \cdot \cos(\theta) = \frac{3\sin(\theta)}{\cos^2(\theta)} - \frac{2\cos(\theta)}{\sin^2(\theta)}
\end{aligned}$$

Setting $L'(\theta) = 0$ and solving for θ could be done in, say, `Maple`, but that is overkill, since a little work can reduce the necessary computation:

$$\begin{aligned}
\frac{3\sin(\theta)}{\cos^2(\theta)} - \frac{2\cos(\theta)}{\sin^2(\theta)} = 0 \quad &\iff \quad \frac{3\sin(\theta)}{\cos^2(\theta)} = \frac{2\cos(\theta)}{\sin^2(\theta)} \quad \iff \quad \frac{\sin^3(\theta)}{\cos^3(\theta)} = \frac{2}{3} \\
&\iff \quad \left(\frac{\sin(\theta)}{\cos(\theta)}\right)^3 = \frac{2}{3} \quad \iff \quad \tan^3(\theta) = \frac{2}{3} \\
&\iff \quad \tan(\theta) = \left(\frac{2}{3}\right)^{1/3} \quad \iff \quad \theta = \arctan\left(\left(\frac{2}{3}\right)^{1/3}\right)
\end{aligned}$$

At this point one can use a decent calculator to find a good approximation to θ. Indulging in just a bit of overkill, the `Maple` command

```
> fsolve(t=arctan((2/3)^(1/3)),t);
```

gives the output:

$$.7180254492$$

(`fsolve` is a relative of `solve` that tries solve equations numerically.) This is, of course, only an approximation, but it should be plenty for our nominally practical purpose. Note that the answer is given in radians by default and that $0 < 0.7180254492 < \frac{\pi}{2}$.

A very good question to ask at this point is whether the critical point we have found is a local maximum, local minimum, or neither. (Recall that maximum length of a pipe that we can carry horizontally around the given corner is the minimum of $L(\theta)$, so we'd really like this critical point to be a minimum.) Since $L(\theta)$ is differentiable wherever it is defined, and tends to infinity as θ approaches 0 or $\frac{\pi}{2}$, the fact that there is just one critical point between 0 and $\frac{\pi}{2}$ means it has to be a minimum, in fact, *the* minimum.

We now plug $\theta = 0.7180254492$ into $L(\theta)$ to find maximum length of a pipe that we can carry horizontally around the given corner. Again, indulging in a bit of overkill, the `Maple` command

```
> evalf(3/cos(0.7180254492)+2/sin(0.7180254492));
```

gives the output:

$$7.023482380$$

(The `evalf` command attempts to find the numerical value of whatever expression it is given.)

Thus the maximum of length of pipe we can carry around the corner at which the two corridors meet is just over 7.02 m. ■