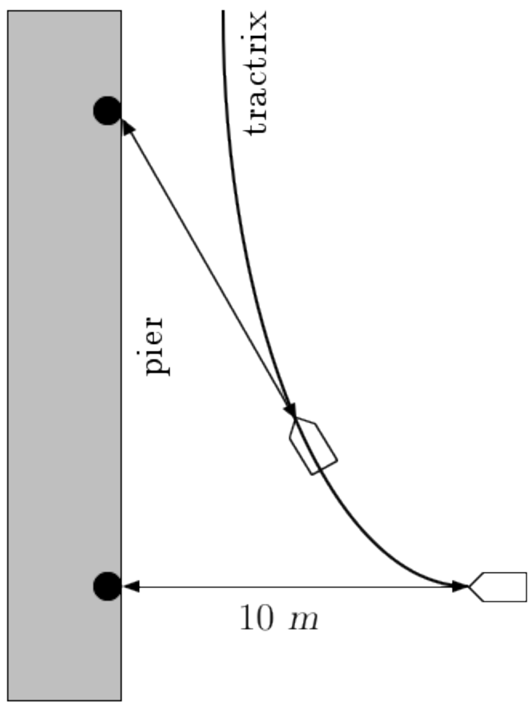
Mathematics 1101Y – Calculus I: functions and calculus of one variable
TRENT UNIVERSITY, 2010–2011

Solutions to Assignment #4
It could have been tractor pulling! :-)

In the beginning, Meredith stands on the edge of a long pier, holding onto one end of a 10 m rope whose other end is attached to the bow of a boat. At this point the rope is stretched out at right angles to the pier. Meredith begins to walk along the edge of the pier while holding onto the rope, thus towing the boat. You may assume that the rope remains straight and taut, and that the path followed by the boat has the property that the rope is always tangent to the curve. (This is an example of a type of curve called a *tractrix.*) See the sketchbelow to help visualize all this.



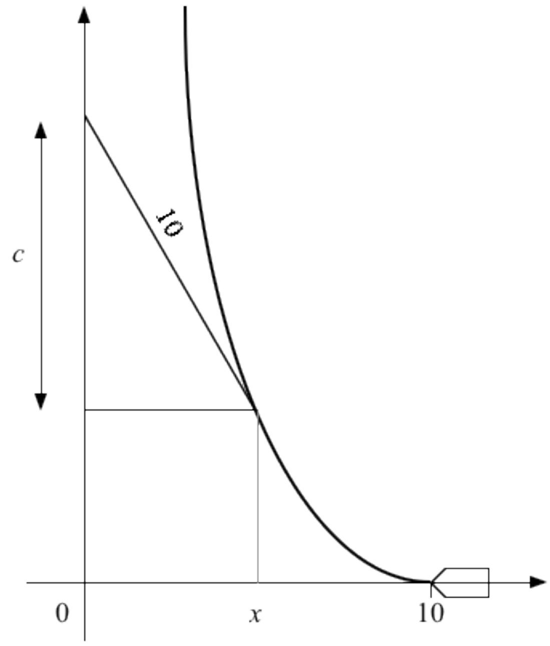
Your task will be to determine just what the path taken by the boat is. To help do this, we'll introduce Cartesian coordinates as follows. Let the y-axis run along the edge of the pier, with the origin at Meredith's starting location, and with the direction Meredith walks in being the positive direction, while the rope is initially stretched out along the positive x-axis. The path followed by the boat will be the graph of $y = f(x)$; note that $f(10) = 0$. We will measure all distances in metres.

We will find the function $f(x)$ in two steps:

1. Determine $\dfrac{dy}{dx} = f'(x)$ in terms of x. *[5]*

Hint: When the boat is at $(x, f(x))$, the rope is still 10 m long and its slope is $\dfrac{dy}{dx} = f'(x)$.

Solution. When the boat is at $(x, f(x))$, consider the right triangle whose hypotenuse is the rope, and hence has length 10, and whose short sides are parallel to the axes.



The base of this triangle, the side parallel to the x-axis, has length $x - 0 = x$; let c be the length of the other short side, the side parallel to the y-axis. By the Pythagorean Theorem, $c^2 + x^2 = 10^2$, so $c = \sqrt{10^2 - x^2} = \sqrt{100 - x^2}$. The slope of the rope – which is equal to $\frac{dy}{dx}$ because the rope is tangent to the curve – is then $\frac{\text{rise}}{\text{run}} = \frac{-c}{x} = -\frac{100 - x^2}{x}$. (Note that the slope must be negative because the rope goes down as we scan from left to right.) It follows that $f'(x) = \frac{dy}{dx} = -\frac{100 - x^2}{x}$. ■

2. Use `Maple` to solve the differential equation you obtained in **1** for $y = f(x)$. *[5]*

Solution. The "Classic Worksheet" `Maple` command

```
> dsolve({diff(y(x),x)=-sqrt(100-x^2)/x,y(10)=0},y(x));
```

gives the result:

$$y(x) = -\sqrt{100 - x^2} + 10\text{arctanh}\left(\frac{10}{\sqrt{100 - x^2}}\right) + 5I\pi$$

The I in the constant term represents the "imaginary" number $i = \sqrt{-1}$. You might ask yourself what it is doing there, given that we are supposed to be getting a real-valued function of the real variable x.

It should be noted that because of the fact that the inverse hyperbolic trig functions are closely related to each other and various logarithm functions – there are lots of identities relating them – $y = f(x)$ can be written in many ways. In fact, the functions giving the tractrix are more often written in terms of arcsech or logarithms, in which case annoying imaginary constants probably won't appear. ■