

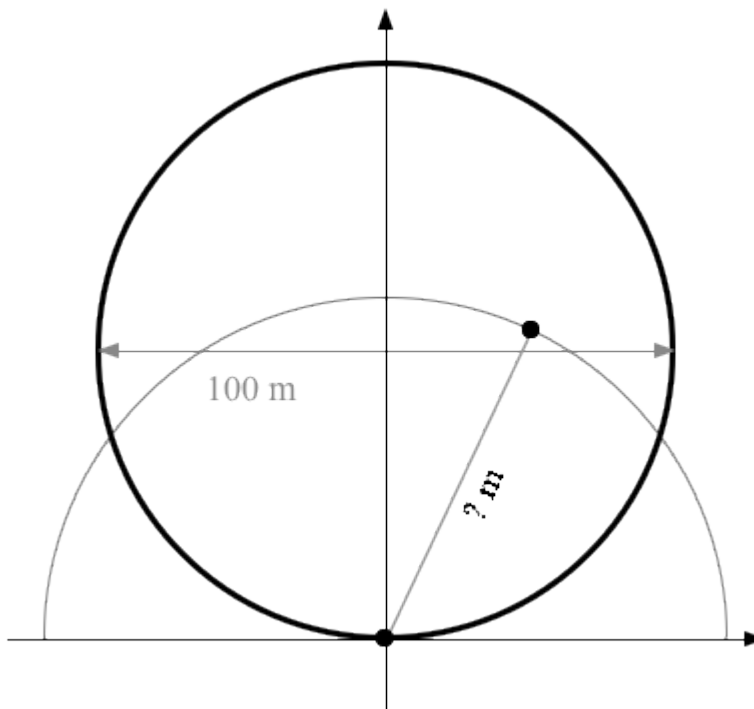
Mathematics 1101Y – Calculus I: functions and calculus of one variable
TRENT UNIVERSITY, 2010–2011

Solution to Assignment #10
The geometric goat

We will attempt to solve the following problem:

A goat is tethered by a rope to a point on the edge of a circular field with diameter 100 meters. What length should the rope be so that the goat can graze in exactly half the field?

Here's a picture, corrected to match the statement of the problem, with some axes added:



Note that it would be equally acceptable to do the problem assuming a radius of 100 meters for the field. In this case the answer ends up being twice what is – eventually – found below.

1. Use polar coordinates to set up and compute an integral – which will involve the unknown length t of the goat's tether – for the area of the part of the field accessible to the goat. [7]

SOLUTION. A math-adventure! We'll do what we can until we get stuck and then try something else if we have to. Just to show you what happens if you tried this in Cartesian coordinates, which most of us are rather more familiar with:

ATTEMPT I. (*Cartesian coordinates*) We'll set up axes as in the diagram above, with the goat tethered at the origin. Note that the field has diameter $100m$ and hence radius $50 m$.

Thus the circular field has equation $x^2 + (y - 50)^2 = 50^2$ and the circle the goat can move around in is given by $x^2 + y^2 = t^2$, where t is the length of the tether. The part of the field the goat can graze in is then the roughly almond-shaped region bounded below by the bottom part of $x^2 + (y - 50)^2 = 50^2$, which has equation $y = 50 - \sqrt{50^2 - x^2}$, and above by the top part of $x^2 + y^2 = t^2$, which has equation $y = \sqrt{t^2 - x^2}$.

To find the area of this region we will also need to know where the two circles intersect. Writing out both equations and subtracting one from the other gives us

$$\begin{aligned} 50^2 - t^2 &= (x^2 + (y - 50)^2) - (x^2 + y^2) \\ &= (x^2 + y^2 - 100y + 50^2) - (x^2 + y^2) = -100y + 50^2, \end{aligned}$$

from which it follows that $100y = t^2$, *i.e.* $y = t^2/100$, at the points of intersection. We get the x values by substituting this into $x^2 + t^2 = t^2$ and solving for x :

$$x^2 + \left(\frac{t^2}{100}\right)^2 = t^2 \implies x = \pm \sqrt{t^2 - \left(\frac{t^2}{100}\right)^2} = \frac{\pm t \sqrt{100^2 - t^2}}{100}$$

The area of the region the goat can graze in (in terms of the length of the tether, r) is then:

$$\begin{aligned} \text{Area} &= \int_{-t\sqrt{100^2-t^2}/100}^{t\sqrt{100^2-t^2}/100} \left(\sqrt{t^2 - x^2} - \left(50 - \sqrt{50^2 - x^2}\right) \right) dx \\ &= \int_{-t\sqrt{100^2-t^2}/100}^{t\sqrt{100^2-t^2}/100} \left(\sqrt{t^2 - x^2} + \sqrt{50^2 - x^2} - 50 \right) dx \\ &\quad \text{(Here I just looked up the antiderivative in the text ... :-)} \\ &= \left[\frac{x}{t} \sqrt{t^2 - x^2} + \frac{t^2}{2} \arcsin\left(\frac{x}{t}\right) + \frac{x}{50} \sqrt{50^2 - x^2} \right. \\ &\quad \left. + \frac{50^2}{2} \arcsin\left(\frac{x}{50}\right) - 50x \right] \Bigg|_{-t\sqrt{100^2-t^2}/100}^{t\sqrt{100^2-t^2}/100} \quad (\text{Ugh!}) \end{aligned}$$

(Note that every part of the antiderivative is an odd function of x .)

$$\begin{aligned} &= 2 \frac{t\sqrt{100^2 - t^2}}{100t} \sqrt{t^2 - \left(\frac{t\sqrt{100^2 - t^2}}{100}\right)^2} + 2 \frac{t^2}{2} \arcsin\left(\frac{t\sqrt{100^2 - t^2}}{100t}\right) \\ &\quad + 2 \frac{t\sqrt{100^2 - t^2}}{100 \cdot 50} \sqrt{50^2 - \left(\frac{t\sqrt{100^2 - t^2}}{100}\right)^2} \\ &\quad + 2 \frac{50^2}{2} \arcsin\left(\frac{t\sqrt{100^2 - t^2}}{100 \cdot 50}\right) - 2 \cdot 50 \frac{t\sqrt{100^2 - t^2}}{100} \end{aligned}$$

Simplify! Simplify! Well, try, and try, and try, and eventually get:

$$\begin{aligned} \text{Area} &= \frac{2t^2\sqrt{100^2 - t^2}}{100^2} + t^2 \arcsin\left(\frac{\sqrt{100^2 - t^2}}{100}\right) \\ &\quad + \frac{2t\sqrt{100^2 - t^2}}{100^2 \cdot 50} \sqrt{50^2 100^2 - 100t^2 + t^4} \\ &\quad + 50^2 \arcsin\left(\frac{t\sqrt{100^2 - t^2}}{100 \cdot 50}\right) - t\sqrt{100^2 - t^2} \end{aligned}$$

Now we have to find the value of t which makes this expression equal to half the area of the field, *i.e.* $\frac{1}{2}\pi 50^2$, which would be a tough job even without the arcsins in the expression. Perhaps we can, after all, do better with polar coordinates, which brings us to:

ATTEMPT II. (*Polar coordinates*) The equation of the field in polar coordinates is $r = 100 \sin(\theta)$, $0 \leq \theta \leq \pi$, and the equation of the circle the goat can roam in is just $r = t$, where t is the length of the tether, as before. To avoid having to break the integral up, we're going to find the area of the part of the field the goat can't reach. This will be

$$\int_{\alpha}^{\pi-\alpha} \frac{1}{2} (100^2 \sin^2(\theta) - t^2) d\theta,$$

where α is the angle for the point to the right of the y -axis where the two circles intersect. It follows from the symmetry about the y -axis of the set-up that the other limit is $\pi - \alpha$.

How do we find α ? We'll use the fact that the relevant point of intersection is a distance of t from the origin and (from our previous attempt) has y -coordinate $y = t^2/100$. Since $y = r \sin(\theta)$ in polar coordinates,

$$\frac{t^2}{100} = t \sin(\alpha) \implies \alpha = \arcsin\left(\frac{t}{100}\right).$$

(Can't seem to get away from those arcsins. Better hope things work out more easily otherwise ...)

Thus, looking up antiderivatives as necessary:

$$\begin{aligned} \text{Area} &= \int_{\alpha}^{\pi-\alpha} \frac{1}{2} (100^2 \sin^2(\theta) - t^2) d\theta \\ &= \frac{1}{2} \left(\frac{1}{2} 100^2 \theta - \frac{1}{4} 100^2 \sin(2\theta) - t^2 \theta \right) \Big|_{\alpha}^{\pi-\alpha} \end{aligned}$$

We plug the limits into the antiderivative and beaver away at trying to simplify the resulting

expression, writing the limits in terms of α as long as possible:

$$\begin{aligned}
&= \frac{1}{2} \left(\frac{1}{2} 100^2 (\pi - \alpha) - \frac{1}{4} 100^2 \sin((2(\pi - \alpha)) - t^2(\pi - \alpha)) \right) \\
&\quad - \frac{1}{2} \left(\frac{1}{2} 100^2 \alpha - \frac{1}{4} 100^2 \sin(2\alpha) - t^2 \alpha \right) \\
&= \frac{1}{2} \left(\frac{1}{2} 100^2 (\pi - 2\alpha) - \frac{1}{4} 100^2 \sin(2\pi - 2\alpha) + \frac{1}{4} 100^2 \sin(2\alpha) - t^2 (\pi - 2\alpha) \right) \\
&= \frac{1}{2} \left(\left(\frac{1}{2} 100^2 - t^2 \right) (\pi - 2\alpha) - \frac{1}{4} 100^2 \sin(-2\alpha) + \frac{1}{4} 100^2 \sin(2\alpha) \right) \\
&= \frac{1}{2} \left(\left(\frac{1}{2} 100^2 - t^2 \right) (\pi - 2\alpha) + \frac{1}{4} 100^2 \sin(2\alpha) + \frac{1}{4} 100^2 \sin(2\alpha) \right) \\
&= \frac{1}{2} \left(\left(\frac{1}{2} 100^2 - t^2 \right) (\pi - 2\alpha) + \frac{1}{2} 100^2 \sin(2\alpha) \right) \\
&= \frac{1}{2} \left(\frac{1}{2} 100^2 - t^2 \right) (\pi - 2\alpha) + \frac{1}{4} 100^2 \sin(2\alpha) \\
&= \frac{1}{2} \left(\frac{1}{2} 100^2 - t^2 \right) (\pi - 2\alpha) + \frac{1}{2} 100^2 \sin(\alpha) \cos(\alpha)
\end{aligned}$$

Now – finally! – we plug in for α , giving

$$\begin{aligned}
\text{Area} &= \frac{1}{2} \left(\frac{1}{2} 100^2 - t^2 \right) \left(\pi - 2 \arcsin \left(\frac{t}{100} \right) \right) + \frac{1}{2} 100^2 \frac{t}{100} \sqrt{1 - \frac{t^2}{100^2}} \\
&= \frac{1}{2} \left(\frac{1}{2} 100^2 - t^2 \right) \left(\pi - 2 \arcsin \left(\frac{t}{100} \right) \right) + \frac{1}{2} t \sqrt{100^2 - t^2}.
\end{aligned}$$

It's a *little* less nasty than the expression we got after working in Cartesian coordinates, but it's still hard to solve for t after setting this equal to $\frac{1}{2}\pi 50^2$... ■

2. Set the expression you obtained in 1 for the area of the part of the field accessible to the goat equal to half the area of the field and solve for the length t of the goat's tether. [3]

SOLUTION. Here's the equation we need to solve, if we can:

$$\begin{aligned}
\frac{1}{2} \pi 50^2 &= \frac{1}{2} \left(\frac{1}{2} 100^2 - t^2 \right) \left(\pi - 2 \arcsin \left(\frac{t}{100} \right) \right) + \frac{1}{2} t \sqrt{100^2 - t^2} \\
\implies \pi 50^2 &= \left(\frac{1}{2} 100^2 - t^2 \right) \left(\pi - 2 \arcsin \left(\frac{t}{100} \right) \right) + t \sqrt{100^2 - t^2}
\end{aligned}$$

It's a mess ... What can we do? Get help! That is, use **Maple**. **Maple** and comparable computer algebra systems, such as **Mathematica**, include tools for finding exact and/or approximate solutions to equations. Your humble author doesn't use these tools very

often, so it took a couple of tries. (Until this school year he got by with a ≥ 12 -year old copy of **Maple V Release 5** before upgrading to the current version of **Maple**, 14. As you will see, the stuff below was tried on **Maple V** first.) “What was the fool command again?”

Searching for “solve” in the Topics submenu of **Maple**’s Help menu showed that it has a `solve` command that would try to solve the supplied equation for the given variable. Unfortunately, on typing in

```
> solve( Pi * 50^2 = (100^2/2 - t^2)*(Pi - 2*arcsin(t/100)) +  
t*sqrt(100^2 - t^2) , t);
```

Maple V appeared to do nothing. Closer inspection of the previously mentioned help item about the `solve` command revealed that it tries to find a symbolic expression for the variable in question and returns nothing at all if it can’t do so. Grrr! Typing the same command into **Maple 14**, in its “Classic Worksheet” mode, gave:

```
100 * sin(RootOf(-4 * _Z + 8 * _Z * cos(_Z)^2 + 3 * Pi - 4 * Pi * cos(_Z)^2 - 4 * sin(_Z) * sqrt(cos(_Z)^2)))
```

This doesn’t seem very useful, especially when you realize that that the `_Z` is probably supposed to be a complex number.

A little further reading about the commands related to `solve`, conveniently listed at the end of the help item for `solve`, revealed that there is a similar command, `fsolve`, that tries to find a numerical solution to a reasonable degree of precision. Typing in

```
> fsolve( Pi * 50^2 = (100^2/2 - t^2)*(Pi - 2*arcsin(t/100)) +  
t*sqrt(100^2 - t^2) , t);
```

gave the output:

57.93642365

Success! Quick sanity check: considering the diagram given at the beginning, we would expect the length of the tether to be something, but not too much, over 50 *m*, which is consistent with a value of *t* close to 58.

Thus the length of the tether needs to be a bit less than 58 *m*. With a field of radius 100 *m* instead of 50 *m*, the tether would need to be a bit less than 116 *m*. ■