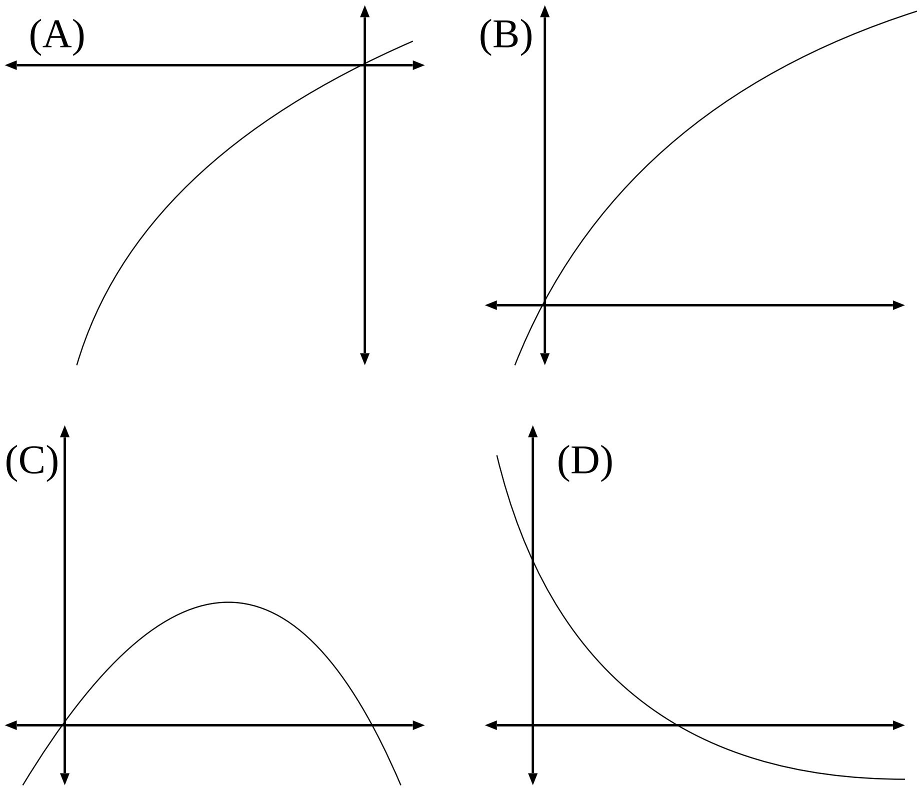
Math 1100 — Calculus, Spring Term Test — 2010-02-26

1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function with the following properties:
   - f is thrice-differentiable, and $f''(x) < 0$ and $f'''(x) > 0$ for all $x \in \mathbb{R}$.
   - $f(0) = 0$ and $f'(0) > 0$.



(10) (a) How many maxima or minima (if any) can f have in the interval $(-\infty, 0)$? Justify your answer.

Solution: f cannot have *any* maxima or minima in the interval $(-\infty, 0)$.

To see this, suppose (by contradiction) that $y < 0$ was a maxium or minimum. Then Fermat's Theorem says that $f'(y) = 0$. But f' is strictly decreasing (because $f'' < 0$). Thus, if $f'(y) = 0$ and $y < 0$, then we would have $f(0) < 0$, which is false. By contradiction, f can't have any maxima/minima in $(-\infty, 0)$.

To see it another way, $f'(y) > 0$ for all $y \in (-\infty, 0)$, so f is strictly increasing in this interval; hence it can't have any extrema here. □

(10) (b) Recall that a point $x \in \mathbb{R}$ is a *zero* of f if $f(x) = 0$. (For example $x = 0$ is a zero of f in this case.) How many zeros (if any) can f have in the interval $(-\infty, 0)$? Justify your answer.

Solution: f cannot have *any* zeros in the interval $(-\infty, 0)$.

To see this, suppose (by contradiction) that $x < 0$ and $f(x) = 0$. Then Rolle's Theorem says there exists some $y \in (x, 0)$ such that $f'(y) = 0$. But as we saw in question (a), this is impossible.

① A lot of people seemed to think that the endpoint 0 was included in the domain $(-\infty, 0)$, and hence, they counted $x = 0$ amongst the 'zeros' in this domain. This is not correct. The interval $(-\infty, 0)$ is *open*. It does not include its endpoints. (In contrast, the interval $(-\infty, 0]$ is *closed*, and *does* include 0). However, this was a minor issue, and I didn't deduct any marks for this as long as the question was otherwise done properly.

A few people also described the point $x = -\infty$ as a 'minimum' of the function. First of all, $-\infty$ is not a real number, so it doesn't count as a 'minimum'. Second of all, even if we did count $-\infty$ as a real number, it would not be included in the *open* interval $(-\infty, 0)$. (If I wanted to include $-\infty$, I would have written $[-\infty, 0)$.) Again, however, I did not deduct marks for this minor confusion.

② Several people wrote something like this: "If f is three times differentiable, that means it is a degree 3 polynomial." They then proceeded to analyse this 'polynomial' —e.g. "a degree 3 polynomial has at most 3 zeros, has at most one maximum and one minimum," etc. This is totally wrong. First of all, there is no reason to believe this function is a polynomial. Second all, any polynomial is *infinitely* differentiable, so information about the first three derivatives tells you nothing about the degree of the polynomial.

□

(10) (c) Sketch the possible graph(s) of f on the interval $(-\infty, 0]$ to illustrate the scenario(s) you claim are possible in parts (a) and (b).

Solution: See Figure A. □

(10) (d) How many maxima or minima (if any) can f have in the interval $(0, \infty)$? Justify your answer.

Solution: f can have *at most one* extreme point in the interval $(0, \infty)$. If it has any extreme point, then it must be a maximum.

To see this, suppose (by contradiction) that $0 < x < z$ are two extreme points. Then Fermat's Theorem implies that $f'(x) = 0 = f'(y)$. But f' is strictly decreasing (because $f'' < 0$), so this is impossible unless $x = y$.

Now, let $x > 0$, and suppose x is an extreme point (so $f'(x) = 0$). Then f must be a maximum, because $f''(x) < 0$ (by hypothesis).

Note: while f *can* have a maximum, it doesn't have to. Figures B and C portray two possibilities.

□

(10) (e) How many zeros (if any) can f have in the interval $(0, \infty)$? Justify your answer.

Solution: f can have *at most one* zero in the interval $(0, \infty)$. To see this, suppose $0 < x < z$ and we have $f(0) = f(x) = f(z) = 0$. Then Rolle's Theorem says there exist some $w \in (0, x)$ such that $f'(w) = 0$, and also some $y \in (x, z)$ such that $f'(y) = 0$. But f' is strictly decreasing (because $f'' < 0$). Thus, we cannot have $f'(w) = 0 = f'(y)$ if $w < y$ —contradiction. By contradiction, f can't have two zeros in $(0, \infty)$.

Note that f *can* have one zero in $(0, \infty)$, but it doesn't have to. Figures B and C portray two possibilities. □

(10) (f) Sketch the possible graph(s) of f on the interval $[0, \infty)$ to illustrate the scenario(s) you claim are possible in parts (d) and (e).

Solution: See Figure B and C. □

(g) Now suppose there is some function $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = \int_0^x g(x)\, dx$ for

(10) all $x \in \mathbb{R}$. What can you say about g' and g''? Where is g increasing/decreasing?

Where is g concave up or concave down? Use this information to sketch a possible graph for g.

Solution: The Fundamental Theorem of Calculus says that $f' = g$. Thus, $g' = f''$ and $g'' = f'''$. Thus, we know that $g'(x) < 0$ and $g''(x) > 0$ for all $x \in \mathbb{R}$ (because $f''(x) < 0$ and $f'''(x) > 0$ for all $x \in \mathbb{R}$). Thus, g is *decreasing* and *concave up* everywhere on $\mathbb{R}$; see Figure D. □

2. Compute the following limits:

(20) (a) $\lim_{x\to\infty} \frac{\ln(x^2+1)}{x}$.

Solution: Let $f(x) = \ln(x^2+1)$ and $g(x) = x$. Then

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(x^2+1)}{x} &= \lim_{x\to\infty} \frac{f(x)}{g(x)} \underset{(*)}{=} \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \\
&\underset{(\dagger)}{=} \lim_{x\to\infty} \frac{2x}{x^2+1} = \lim_{x\to\infty} \frac{2/x}{1+1/x^2} \\
&= \frac{\lim_{x\to\infty}(2/x)}{\lim_{x\to\infty}(1+1/x^2)} = \frac{0}{1} = \boxed{0.}
\end{aligned}$$

Here, $(*)$ is by l'Hospital's rule, which is applicable because $\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x)$. Next, $(\dagger)$ is because $f'(x) = \frac{2x}{x^2+1}$ and $g'(x) = 1$. □

(10) (b) $\lim_{x\to\infty} (x^2+1)^{1/x}$.

Solution: This is an indeterminate form of type "∞^0". We take the logarithm, and find that

$$\begin{aligned}
\ln\left(\lim_{x\to\infty} (x^2+1)^{1/x}\right) &= \lim_{x\to\infty} \ln\left((x^2+1)^{1/x}\right) = \lim_{x\to\infty} \frac{1}{x}\ln(x^2+1) \\
&= 0,
\end{aligned}$$

where the last step is by part (a). Thus, $\lim_{x\to\infty} (x^2+1)^{1/x} = e^0 = \boxed{1.}$ □

3. Compute the following integrals:

(25) (a) $\int \sin(x)^3 \cos(x)^5 \, dx$.

Solution:

$$\begin{aligned}
\int \sin(x)^3\cos(x)^5\,dx &= \int \cos(x)^5 \cdot \sin(x)^2 \cdot \sin(x)\,dx \underset{(*)}{=} \int \cos(x)^5 \cdot (1-\cos(x)^2)\cdot \sin(x)\,dx \\
&\underset{(\dagger)}{=} -\int u^5 \cdot (1-u^2)\,du = -\int u^5 - u^7\,du = -\frac{u^6}{6} + \frac{u^8}{8} + C \\
&\underset{(\dagger)}{=} \boxed{\frac{\cos(x)^8}{8} - \frac{\cos(x)^6}{6} + C.}
\end{aligned}$$

Here $(*)$ is by Pythagoras' equation $\sin(x)^2 + \cos(x)^2 = 1$. Next $(\dagger)$ is the substitution $u := \cos(x)$ so that $du = -\sin(x)\,dx$. □

(25) (b) $\int \frac{u}{\sqrt{1+u^2}}\,du$.

Solution: Let $y := 1 + u^2$; then $dy = 2u\,du$, so $u\ du = \frac{1}{2}\ dy$. Thus

$$\int \frac{u\ du}{\sqrt{1+u^2}} \quad = \quad \frac{1}{2}\int \frac{dy}{\sqrt{y}}\ dy \quad = \quad \frac{1}{2}\int y^{-1/2}\ dy \quad = \quad y^{1/2} + C \quad = \quad \boxed{(1+u^2)^{1/2}\ +\ C.}$$

□

Solution: Another approach uses a 'trig substitution'. Let $u := \tan(\theta)$; then $du = \sec(\theta)^2\ d\theta$. Meanwhile,

$$\sqrt{1+u^2} \quad = \quad \sqrt{1+\tan(\theta)^2} \quad = \quad \sqrt{\sec(\theta)^2} \quad = \quad |\sec(\theta)| \quad = \quad \sec(\theta),$$

where the last step assumes $-\pi/2 < \theta < \pi/2$. Substituting this all in, we get

$$\int \frac{u}{\sqrt{1+u^2}}\ du \quad = \quad \int \frac{\tan(\theta)}{\sec(\theta)} \cdot \sec(\theta)^2\ d\theta \quad = \quad \int \tan(\theta)\sec(\theta)\ d\theta$$
$$= \quad \sec(\theta) + C \quad = \quad \sec(\arctan(u)) + C \quad = \quad \sqrt{1+u^2} + C.$$

where the last step follows from a 'Pythagoras triangle' argument. □

(25) (c) $\displaystyle\int \frac{\ln(x)}{x\cdot\sqrt{1+\ln(x)^2}}\ dx.$

Solution: Let $u := \ln(x)$. Then $du = \frac{1}{x}\ dx$. Thus,

$$\int \frac{\ln(x)}{x\cdot\sqrt{1+\ln(x)^2}}\ dx \quad = \quad \int \frac{u}{\sqrt{1+u^2}}\ du \underset{(*)}{=} (1+u^2)^{1/2} + C \quad = \quad \boxed{\sqrt{1+\ln(x)^2}\ +\ C,}$$

here $(*)$ is by question (b). □

(25) (d) $\displaystyle\int x\cdot e^{-x}\ dx.$

Solution: We will use integration by parts. Let $u := x$, so that $du = dx$. Let $dv := e^{-x}\ dx$; then $v = -e^{-x}$. Thus,

$$\int x\cdot e^{-x}\ dx \quad = \quad \int u\ dv \quad = \quad uv - \int v\ du$$
$$= \quad -xe^{-x} - \int -e^{-x}\ dx \quad = \quad -xe^{-x} - e^{-x} + C$$
$$= \quad \boxed{-e^{-x}\cdot(x+1) + C.}$$

□

Common minor mistakes: A lot of people forgot to add the constant term "$+C$" to the indefinite integrals. This cost 2 marks (out of 25) per question.

Also, a lot of people forgot to 'reverse' their substitutions (e.g. in question #3(b), they would leave $\sqrt{y} + C$ as a final answer). This cost 5 marks (out of 25) per question.

Finally, some divided or multiplied by the wrong constant when antidifferentiation. For example in question #3(b), they would end up with $\frac{1}{2}\sqrt{1+u^2} + C$ or $2\sqrt{1+u^2} + C$ as a final answer. This cost 5 marks (out of 25) per question.

Major mistakes: Some people tried to *differentiate* instead of antidifferentiating (e.g. in question #3(a) they applied the Leibniz rule to differentiate $\sin(x)^3\cos(x)^5$). Also, some people tried to 'factor' the integral (e.g. they wrote "$\int \sin(x)^3\cos(x)^5\ dx = \int \sin(x)^3\ dx \cdot \int \cos(x)^5\ dx$", or at least, antidifferentiated each term separately, as if this was the case). This is totally wrong.