Math 1100 — Calculus, Quiz #9A — 2009-12-07

Let $f(x) := \dfrac{1}{1+\exp(-x)} - \dfrac{1}{2}$, for all $x \in \mathbb{R}$ where this formula makes sense.

(5) 1. What is the domain of f?

Solution: The domain is $\mathbb{R}$, because the formula for f makes sense for all $x \in \mathbb{R}$ (the denominator is never zero, because $\exp(-x) > 0$ for all $x \in \mathbb{R}$). □

(10) 2. What are the x-intercepts and y-intercepts of f?

Solution: For all $x \in \mathbb{R}$,

$$\Big(f(x) = 0\Big) \iff \left(\frac{1}{1+\exp(-x)} = \frac{1}{2}\right) \iff \Big(2 = 1 + \exp(-x)\Big)$$
$$\iff \Big(1 = \exp(-x)\Big) \iff \Big(x = 0\Big).$$

Thus, $\boxed{\text{the only } x\text{-intercept (and also, } y\text{-intercept) is at } (0,0).}$ □

(15) 3. What symmetries does f have? Is it odd? even? Periodic?

Solution: The function is *odd*,. To see this, observe that

$$f(x) + f(-x) = \frac{1}{1+e^{-x}} - \frac{1}{2} + \frac{1}{1+e^{x}} - \frac{1}{2} = \frac{(1+e^x) + (1+e^{-x})}{(1+e^{-x})(1+e^x)} - 1$$
$$= \frac{2 + e^x + e^{-x}}{1 + e^{-x} + e^x + 1} - 1 = \frac{2 + e^x + e^{-x}}{2 + e^{-x} + e^x} - 1 = 1 - 1 = 0.$$

and thus, $f(-x) = -f(x)$. □

(15) 4. Find all vertical, horizontal, and slant asymptotes of f.

Solution: There are no vertical asymptotes because $f(x)$ is finite for all $x \in \mathbb{R}$. As for horizontal asymptotes, we have:

$$\lim_{x\to-\infty} \frac{1}{1-\exp(-x)} - \frac{1}{2} = \frac{1}{1 - \lim_{x\to-\infty}\exp(-x)} - \frac{1}{2}$$
$$= \frac{1}{1 - \lim_{x\to\infty}\exp(x)} - \frac{1}{2}$$
$$= 0 - \frac{1}{2} = \boxed{-\frac{1}{2}}.$$

and
$$\lim_{x\to\infty} \frac{1}{1-\exp(-x)} - \frac{1}{2} = \frac{1}{1 - \lim_{x\to\infty}\exp(-x)} - \frac{1}{2} = \frac{1}{1-0} - \frac{1}{2}$$
$$= 1 - \frac{1}{2} = \boxed{\frac{1}{2}}.$$

□

(15) 5. Compute f'. Use this to find all intervals where f is increasing/decreasing.

Solution: The quotient rule says $f'(x) = \dfrac{\exp(-x)\cdot 0 - 1\cdot(-\exp(-x))}{(1-\exp(-x))^2} = \dfrac{\exp(-x)}{(1-\exp(-x))^2}$.

The denominator of f' is always positive (because it is a squared expression). The numerator is also always positive (because $\exp(x) > 0$ for all $x \in \mathbb{R}$). Thus, $f'(x)$ is always positive, so $\boxed{f \text{ is always increasing.}}$ □

(10) 6. Find all local maxima and minima of f.

Solution: f has no critical points, because f' is always defined and never zero. Thus, f has no maxima or minima. □

(15) 7. Compute f''. Use this to identify the intervals of concavity and inflection points.

Solution: We have seen that

$$f'(x) = \frac{e^{-x}}{(1-e^{-x})^2} = \frac{e^{-x}}{1-2e^{-x}+e^{-2x}} = \frac{1}{e^x - 2 + e^{-x}}$$

$$\text{Thus,}\quad f''(x) = \frac{(e^x - 2 + e^{-x})\cdot 0 - 1\cdot(e^x - e^{-x})}{(e^x-2+e^{-x})^2} = \frac{e^{-x}-e^x}{(e^x-2+e^{-x})^2}$$

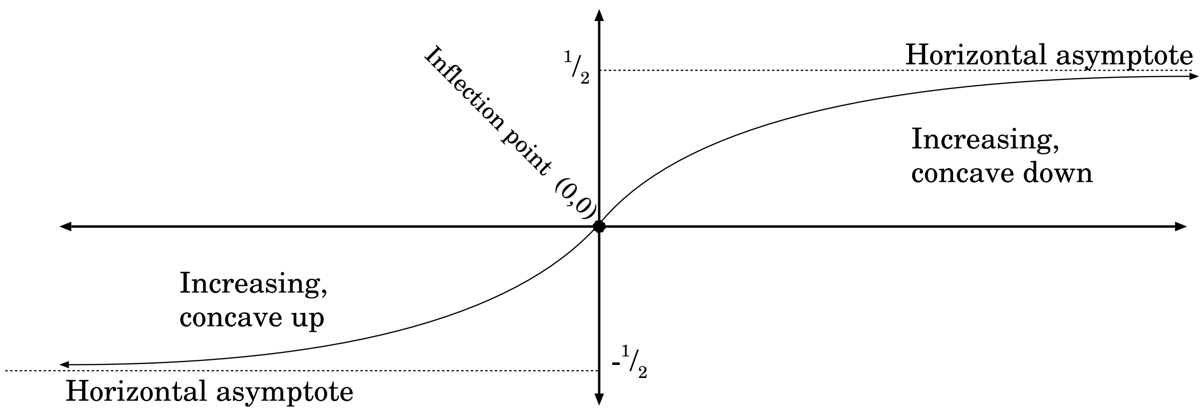
The denominator of $f''(x)$ is always positive (it is a squared expression). Thus

$$\Big(f''(x) > 0\Big) \iff \Big(e^{-x} - e^x > 0\Big) \iff \Big(e^{-x} > e^x\Big) \iff \Big(e^{-2x} > 1\Big) \iff \Big(-2x > 0\Big) \iff \Big(x < 0\Big).$$

Likewise, $\Big(f''(x) < 0\Big) \iff \Big(x > 0\Big)$. Thus, f is $\boxed{\text{concave up on } (-\infty, 0),}$ f is $\boxed{\text{concave down on } (0,\infty),}$ and f has an $\boxed{\text{inflection point at } 0.}$ □

(15) 8. Use all of the above information to sketch the curve of f on its domain.

Solution:



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