Math 1100 — Calculus, Quiz #3B — 2009-10-09

Let $f(x) := x^2 + x - 4$. We will show that f is continuous at the point $a = 3$, by using the "ϵ, δ" definition of limits.

(10) 1. Factor the polynomial $x^2 + x - 12$.

Solution: $x^2 + x - 12 = (x+4)(x-3)$. □

(20) 2. Suppose $|x - 3| < 1$. Show that $|f(x) - 8| < 8 \cdot |x - 3|$.

Solution: $f(x) - 8 = (x^2 + x - 4) - 8 = x^2 + x - 12 = (x+4)(x-3)$, the polynomial we factored in #1. If $|x - 3| < 1$, then $2 < x < 4$. Thus, $6 < x + 4 < 8$. Thus, $|x + 4| < 8$. Thus,

$$|f(x) - 8| \quad = \quad |x^2 + x - 12| \underset{(*)}{=} |x+4| \cdot |x-3| \quad \leq \quad 8 \cdot |x - 3|.$$

where $(*)$ is by #1. □

(40) 3. Let $\epsilon > 0$. Give a procedure to construct $\delta > 0$ such that, for any $x \in \mathbb{R}$, we have:

$$\Big(|x - 3| < \delta\Big) \Longrightarrow \Big(|f(x) - 8| < \epsilon\Big). \qquad (1)$$

Solution: Let $\delta := \min\{1, \epsilon/8\}$. Let $x \in \mathbb{R}$. If $|x - 3| < \delta$, then $|x - 3| < 1$ (because $\delta \leq 1$). Thus, we have:

$$|f(x) - 8| \underset{(*)}{<} 8 \cdot |x - 3| \underset{(\dagger)}{<} 8 \cdot \delta \underset{(\ddagger)}{\leq} \epsilon.$$

Here, $(*)$ is by #2, $(\dagger)$ is because $|x - 3| < \delta$, and $(\ddagger)$ is because $\delta \leq \epsilon/8$.

Thus, we conclude that $|f(x) - 8| < \epsilon$, as desired. □

(10) 4. Explain how #3 implies that f is continuous at 3.

Solution: We have shown that, for any $\epsilon > 0$, we can construct a $\delta > 0$ such that statement (1) holds. This means that $\lim_{x \to 3} f(x) = 8$. But $f(3) = 9 + 3 - 4 = 8$. Thus, f is continuous at 3. □

(20) 5. In fact, f is continuous everywhere on $\mathbb{R}$ (this can be shown by generalizing the above proof). Using this fact, show that there exists some $x \in [0, 2]$ such that $f(x) = 0$.

Solution: We have $f(0) = -4 < 0 < 2 = f(2)$. Thus, the Intermediate Value Theorem implies that there exists some $x \in [0, 2]$ such that $f(x) = 0$. □