Math 1100 — Calculus, Quiz #3A — 2009-10-05

Let $f(x) := x^2$. We will show that f is continuous at the point $a = 5$, by using the "ϵ, δ" definition of limits.

(10) 1. Factor the polynomial $x^2 - 25$.

Solution: $x^2 - 25 = (x+5)(x-5)$. □

(20) 2. Suppose $|x-5| < 1$. Show that $|x^2 - 25| < 11 \cdot |x-5|$.

Solution: If $|x-5| < 1$, then $4 < x < 6$. Thus, $9 < x+5 < 11$. Thus, $|x+5| < 11$. Thus,

$$|x^2 - 25| \underset{(*)}{=} |x+5| \cdot |x-5| \leq 11 \cdot |x-5|.$$

where $(*)$ is by #1. □

(40) 3. Let $\epsilon > 0$. Give a procedure to construct $\delta > 0$ such that, for any $x \in \mathbb{R}$, we have:

$$\Big(|x-5| < \delta\Big) \Longrightarrow \Big(|x^2 - 25| < \epsilon\Big). \quad (1)$$

Solution: Let $\delta := \min\{1, \epsilon/11\}$. Let $x \in \mathbb{R}$. If $|x-5| < \delta$, then $|x-5| < 1$ (because $\delta \leq 1$). Thus, we have:

$$|x^2 - 25| \underset{(*)}{<} 11 \cdot |x-5| \underset{(\dagger)}{<} 11 \cdot \delta \underset{(\ddagger)}{\leq} \epsilon.$$

Here, $(*)$ is by #2, $(\dagger)$ is because $|x-5| < \delta$, and $(\ddagger)$ is because $\delta \leq \epsilon/11$.

Thus, we conclude that $|x^2 - 25| < \epsilon$, as desired. □

(10) 4. Explain how #3 implies that f is continuous at 5.

Solution: We have shown that, for any $\epsilon > 0$, we can construct a $\delta > 0$ such that statement (1) holds. This means that $\lim_{x \to 5} f(x) = 25$. But $f(5) = 25$. Thus, f is continuous at 5. □

(20) 5. In fact, f is continuous everywhere on $\mathbb{R}$ (this can be shown by generalizing the above proof). Using this fact, show that there exists some $x \in [5, 6]$ such that $f(x) = 30$.

Solution: We have $f(5) = 25 < 30 < 36 = f(6)$. Thus, the Intermediate Value Theorem implies that there exists some $x \in [5, 6]$ such that $f(x) = 30$. □