Math 1100 — Calculus, Quiz #16B — 2010-03-18

The following formulae might be useful:

$$\begin{aligned} \int \tan(x)\ dx &= \ln|\sec(x)| + C. & \int \sec(x)\ dx &= \ln|\sec(x)+\tan(x)| + C. \\ \int \cot(x)\ dx &= \ln|\sin(x)| + C. & \int \csc(x)\ dx &= \ln|\csc(x)-\cot(x)| + C. \\ \int \sec(x)^2\ dx &= \tan(x) + C. & \int \sec(x)\tan(x)\ dx &= \sec(x) + C. \\ \int \csc(x)^2\ dx &= -\cot(x) + C. & \int \csc(x)\cot(x)\ dx &= -\csc(x) + C. \end{aligned}$$

(50) 1. Let $f(x) = \ln[\cos(x)]$. Compute the *arc-length* of the graph of f between the points $x = 0$ and $x = \pi/4$.

Solution: If $f(x) = \ln[\cos(x)]$, then

$$\begin{aligned} f'(x) &= \frac{\cos'(x)}{\cos(x)} = \frac{-\sin(x)}{\cos(x)} = -\tan(x). \\ \text{Thus,}\quad \sqrt{1+f'(x)^2} &= \sqrt{1+\tan(x)^2} = \sqrt{\sec(x)^2} = |\sec(x)| \underset{(*)}{=} \sec(x). \\ \text{Thus,}\quad \text{arc-length} &= \int_0^{\pi/4} \sqrt{1+f'(x)^2}\ dx = \int_0^{\pi/4} \sec(x)\ dx \\ &= \ln|\sec(x)+\tan(x)|\ \Big|_{x=0}^{x=\pi/4} \\ &= \ln|\sec(\pi/4)+\tan(\pi/4)| - \ln|\sec(0)+\tan(0)| \\ &= \ln\left|\sqrt{2}+1\right| - \ln|1+0| = \boxed{\ln\left(\sqrt{2}+1\right).} \end{aligned}$$

Here $(*)$ uses the fact that $\sec(\theta) = 1/\cos(\theta) > 0$ for $\theta \in [0, \pi/4]$. □

(50) 2. Let $\mathcal{C}$ be the graph of the function $f(x) = \sin(x)$ for $x \in [0, \pi]$. Let $\mathcal{S}$ be the surface of revolution obtained by rotating $\mathcal{C}$ around the x axis. Compute the *surface area* of $\mathcal{S}$.

Solution: If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$. Thus,

$$\begin{aligned} \text{Area} &= 2\pi \int_0^{\pi} f(x)\sqrt{1+f'(x)^2}\ dx = 2\pi \int_0^{\pi} \sin(x)\sqrt{1+\cos(x)^2}\ dx \\ &\underset{(*)}{=} -2\pi \int_1^{-1} \sqrt{1+u^2}\ du = 2\pi \int_{-1}^{1} \sqrt{1+u^2}\ du \underset{(\dagger)}{=} 2\pi \int_{-\pi/4}^{\pi/4} \sqrt{1+\tan(\theta)^2}\ \sec(\theta)^2\ d\theta \\ &= 2\pi \int_{-\pi/4}^{\pi/4} \sqrt{\sec(\theta)^2}\ \sec(\theta)^2\ d\theta = 2\pi \int_{-\pi/4}^{\pi/4} |\sec(\theta)|\ \sec(\theta)^2\ d\theta \\ &\underset{(\ddagger)}{=} 2\pi \int_{-\pi/4}^{\pi/4} \sec(\theta)\ \sec(\theta)^2\ d\theta = 2\pi \int_{-\pi/4}^{\pi/4} \sec(\theta)^3\ d\theta \\ &\underset{(\diamond)}{=} \frac{2\pi}{2}\left(\sec(\theta)\tan(\theta) + \ln|\sec(\theta)+\tan(\theta)|\right)_{\theta=-\pi/4}^{\theta=\pi/4} \end{aligned}$$

$$
\begin{aligned}
&= \pi \left(\sec\left(\frac{\pi}{4}\right) \tan\left(\frac{\pi}{4}\right) - \sec\left(\frac{\pi}{4}\right) \tan\left(\frac{-\pi}{4}\right) \right. \\
&\qquad\qquad + \quad \left. \ln\left|\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right| - \ln\left|\sec\left(\frac{-\pi}{4}\right) + \tan\left(\frac{-\pi}{4}\right)\right| \right) \\
&= \pi \left(\sqrt{2} \cdot 1 - \sqrt{2} \cdot (-1) + \ln\left|\sqrt{2}+1\right| - \ln\left|\sqrt{2}-1\right| \right) \\
&= \boxed{\pi \left(2\sqrt{2} + \ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right| \right).}
\end{aligned}
$$

Here $(*)$ is the change of variables $u := \cos(x)$ so that $\mathrm{d}u = -\sin(x)\ \mathrm{d}x$. Also, $\cos(0) = 1$ and $\cos(\pi) = -1$.

Next, $(\dagger)$ is the change of variables $u := \tan(\theta)$ so that $\mathrm{d}u = \sec^2(\theta)\ \mathrm{d}\theta$. Also, $\tan\left(\frac{\pi}{4}\right) = 1$ and $\tan\left(\frac{-\pi}{4}\right) = -1$.

Next, $(\ddagger)$ uses the fact that $\sec(x) \geq 0$ for all $x \in \left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$.

Finally, equality $(\diamond)$ was Example 8 on pages 464-465 of section 7.2 in the text. $\square$