Math 1100 — Calculus, Quiz #15B — 2010-03-12

Determine whether each of the following improper integrals is convergent or divergent. If it is convergent, then compute its exact value.

(50) 1. $\displaystyle\int_e^\infty \frac{1}{x(\ln(x))^3}\, dx.$

Solution: Let $u = \ln(x)$. Then $du = \frac{1}{x}\, dx$. thus,

$$\int_e^\infty \frac{1}{x(\ln(x))^3}\, dx = \int_1^\infty \frac{1}{u^3}\, du = \lim_{b\to\infty} \int_1^b u^{-3}\, du = \lim_{b\to\infty} \frac{-u^{-2}}{2}\Big|_{u=1}^{u=b} = \lim_{b\to\infty} \frac{-b^{-2}+1}{2} = \boxed{\frac{1}{2}.}$$

□

(50) 2. $\displaystyle\int_e^\infty \frac{1+\cos(x)^2}{x\ln(x)}\, dx.$

Solution: This integral is $\boxed{\text{divergent.}}$ To see this, observe that $1+\cos(x)^2 \geq 1$ for all $x \in \mathbb{R}$. Thus,

$$\frac{1+\cos(x)^2}{x\ln(x)} \geq \frac{1}{x\ln(x)}$$

for all $x \in \mathbb{R}$. However,

$$\int_e^\infty \frac{1}{x\ln(x)}\, dx \underset{(*)}{=} \int_1^\infty \frac{1}{u}\, du \underset{(\dagger)}{=} \infty.$$

Here, $(*)$ is the substitution $u = \ln(x)$ so that $du = \frac{1}{x}\, dx$, and $(\dagger)$ is an example we did in class (or you can observe that $\int_1^\infty \frac{1}{u}\, du = \lim_{b\to\infty} \ln(b) = \infty$).

Thus, $\displaystyle\int_1^\infty \frac{1+\cos(x)^2}{\ln(x)}\, dx = \infty$, by the Comparison Test. □