Math 1100 — Calculus, Final Exam — 2010-04-12

1. Recall that $\arctan'(x) = \dfrac{1}{1+x^2}$.

$(\frac{5}{200})$ (a) Use this fact to obtain a MacLaurin series for $\arctan(x)$.

Solution: The Geometric Series formula tells us that

$$\frac{1}{1+x^2} \quad = \quad \frac{1}{1-(-x^2)} \quad = \quad \sum_{n=0}^{\infty}(-x^2)^n \quad = \quad \sum_{n=0}^{\infty}(-1)^n\, x^{2n},$$

for all $x \in (-1,1)$. But $\arctan'(x) = \dfrac{1}{1+x^2}$. Thus, the Fundamental Theorem of Calculus says that

$$\begin{aligned} C + \arctan(x) \quad &= \quad \int \arctan'(x)\ \mathrm{d}x \quad = \quad \int \frac{1}{1+x^2}\ \mathrm{d}x \quad = \quad \int \sum_{n=0}^{\infty}(-1)^n\, x^{2n}\ \mathrm{d}x \\ &= \quad \sum_{n=0}^{\infty}(-1)^n \int x^{2n}\ \mathrm{d}x \quad = \quad \sum_{n=0}^{\infty}\frac{(-1)^n\, x^{2n+1}}{2n+1}, \end{aligned}$$

for all $x \in (-1,1)$, where C is some constant. Setting $x = 0$, we get

$$0 \quad = \quad \sum_{n=0}^{\infty}\frac{(-1)^n\, 0^{2n+1}}{2n+1} \quad = \quad C + \arctan(0) \quad = \quad C + 0,$$

so we conclude that $C = 0$. Thus, $\boxed{\arctan(x) \quad = \quad \sum_{n=0}^{\infty}\frac{(-1)^n\, x^{2n+1}}{2n+1}.}$

Common mistakes and remarks on grading: Some people set up the right power series, but wrote "$\sum_{n=1}^{\infty}$" (i.e. they forgot the $n = 0$ term). This got 4/5 marks

Some people tried to compute the MacLaurin series 'the hard way', by explicitly computing $\arctan'(0)$, $\arctan''(0)$, $\arctan'''(0)$, $\arctan^{(4)}(0)$, etc. and looking for a pattern. It is pretty much impossible to make this approach work out. These people got 2/5 for heroic effort.

□

$(\frac{5}{200})$ (b) Now obtain a MacLaurin series for $f(y) := \dfrac{\arctan(y^3)}{y^2}$.

Solution: Let $x := y^3$. Then $x \in (-1,1)$ if and only if $y \in (-1,1)$. If we substitute $x = y^3$ into the MacLaurin series in part (a), we get

$$\arctan(y^3) \quad = \quad \sum_{n=0}^{\infty}\frac{(-1)^n\,(y^3)^{2n+1}}{2n+1} \quad = \quad \sum_{n=0}^{\infty}\frac{(-1)^n\, y^{6n+3}}{2n+1}, \qquad \text{for all } y \in (-1,1).$$

Now we divide each term by y^2 to obtain:

$$\frac{\arctan(y^3)}{y^2} \quad = \quad \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n\, y^{6n+1}}{2n+1},} \qquad \text{for all } y \in (-1,1).$$

□

$(\frac{5}{200})$ (c) Compute $f^{(61)}(0)$ (i.e. the 61st derivative of f at zero).

Solution: Let a_{61} be the 61st coefficient in the MacLaurin serie from part (b). Then a_{61} is the coefficient of y^{6n+1} where $6n+1 = 61$ —hence $n = 10$, hence $a_{61} = \frac{(-1)^{10}}{2 \cdot 10 + 1} = \frac{1}{21}$. Thus,

$$f^{(61)}(0) \quad = \quad 61! \cdot a_{61} \quad = \quad \boxed{\frac{61!}{21}.}$$

Common mistakes and remarks on grading: Some people substituted $n = 61$ instead of $n = 10$, and ended up with $\frac{61!}{123}$. This got 3/5 marks.

□

$(\frac{5}{200})$ (d) Now obtain a MacLaurin series for the general antiderivative $\displaystyle\int \frac{\arctan(x^3)}{x^2}\, \mathrm{d}x$.

Solution: We antidifferentiate the power series in part (b):

$$\begin{aligned}
\int \frac{\arctan(x^3)}{x^2}\, \mathrm{d}x \quad &= \quad \int \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{6n+1}}{2n+1}\, \mathrm{d}x \quad = \quad \sum_{n=0}^{\infty} \int \frac{(-1)^n\, x^{6n+1}}{2n+1}\, \mathrm{d}x \\
&= \quad \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{6n+2}}{(2n+1)(6n+2)} + C,}
\end{aligned}$$

where C is an arbitrary constant. □

$(\frac{10}{200})$ (e) Use the Ratio Test to determine the radius of convergence of the MacLaurin series you derived in part (d).

Solution: Fix $x \in \mathbb{R}$, and define $a_n := \frac{(-1)^n\, x^{6n+2}}{(2n+1)(6n+2)}$. Then

$$\begin{aligned}
\frac{|a_{n+1}|}{|a_n|} \quad &= \quad \frac{\dfrac{|x^{6(n+1)+2}|}{(2(n+1)+1)(6(n+1)+2)}}{\dfrac{|x^{6n+2}|}{(2n+1)(6n+2)}} \quad = \quad \frac{|x^{6n+8}|}{|x^{6n+2}|} \cdot \frac{(2n+1)(6n+2)}{(2n+3)(6n+8)} \\
&= \quad |x|^6 \cdot \frac{(2n+1)(6n+2)}{(2n+3)(6n+8)}.
\end{aligned}$$

Thus, $$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \quad &= \quad \lim_{n\to\infty} |x|^6 \cdot \frac{(2n+1)(6n+2)}{(2n+3)(6n+8)} \quad = \quad |x|^6 \cdot \lim_{n\to\infty} \frac{(2n+1)(6n+2)}{(2n+3)(6n+8)} \\
&= \quad |x|^6 \cdot \left(\lim_{n\to\infty} \frac{2n+1}{2n+3}\right) \cdot \left(\lim_{n\to\infty} \frac{6n+2}{6n+8}\right) \quad = \quad |x|^6.
\end{aligned}$$

Thus,

$$\left(\frac{|a_{n+1}|}{|a_n|} < 1\right) \iff \left(|x|^6 < 1\right) \iff \left(|x| < 1\right).$$

Thus, the radius of convergence is $\boxed{R = 1}$.

Common mistakes and remarks on grading: A lot of people correctly set up the Ratio test, but then screwed up the execution and ended up with the wrong answer. This generally got 2/10 marks. □

2. Determine whether the following series are divergent, conditionally convergent, or absolutely convergent.

$(\frac{15}{200})$ (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{\sqrt{n^2+1}}$.

Solution: This series is $\boxed{\text{divergent.}}$ To see this, observe that

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} \quad = \quad \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n^2}} \quad = \quad 1.$$

Thus, the summands to not converge to zero. The series must diverge. □

$(\frac{15}{200})$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{\sqrt{n^4+1}}$.

Solution: This series is $\boxed{\text{conditionally convergent.}}$ To see this, observe that the sequence $\left\{\dfrac{n}{\sqrt{n^4+1}}\right\}_{n=1}^{\infty}$ is decreasing. Also,

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^4+1}} \quad = \quad \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1/n^2}} \quad = \quad 0.$$

Thus, the Alternating Series Test says the series converges. However, the series does *not* converge absolutely. To see this, we use the Ratio Comparison Test to compare the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+1}}$ to the (divergent) series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. We have

$$\lim_{n\to\infty} \frac{1/n}{n/\sqrt{n^4+1}} \quad = \quad \lim_{n\to\infty} \frac{\sqrt{n^4+1}}{n^2} \quad = \quad \lim_{n\to\infty} \sqrt{1+1/n^4} \quad = \quad 1.$$

Thus, as the series series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+1}}$ also diverges. Thus, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{\sqrt{n^4+1}}$ is *not* absolutely convergent.

Common mistakes and remarks on grading: People got 8/10 if they showed the series converged conditionally (using alternating series test), but failed to address the issue of absolute convergence. Likewise, people got 8/10 if they showed that the series does *not* converge absolutely, but failed to show that it converges conditionally.

Several people attempted to use the Ratio Test to answer #2(a) and #2(b). The Ratio Test is inconclusive in this case (the limit ratio is 1), so this is not a good strategy. These people got 4 marks for a correct attempt, however. □

3. Compute the following integrals:

$(\frac{15}{200})$ (a) $\displaystyle\int \frac{x+12}{(x+5)(x-2)}\, dx$.

Solution: We wish to find constants A and B such that

$$\frac{x+12}{(x+5)(x-2)} \quad = \quad \frac{A}{x+5}+\frac{B}{x-2} \quad = \quad \frac{A(x-2)+B(x+5)}{(x+5)(x-2)} \quad = \quad \frac{(A+B)\,x+(5B-2A)}{(x+5)(x-2)}.$$

Thus, we need $(A+B)x+(5B-2A) = x+12$, which is equivalent to the system of linear equations

$$\begin{aligned} A+B &= 1; && (1)\\ 5B-2A &= 12. && (2) \end{aligned}$$

Adding 2 times equation (1) to equation (2), we get

$$7B + 0A \quad = \quad 14.$$

Thus, $B = 2$. Substituting this into equation (1), we get $A + 2 = 1$; hence $A = -1$. Putting it together, we have

$$\frac{x+12}{(x+5)(x-2)} \quad = \quad \frac{-1}{x+5} + \frac{2}{x-2}.$$

Thus,
$$\int \frac{x+12}{(x+5)(x-2)}\,\mathrm{d}x \quad = \quad \int \frac{-1}{x+5} + \frac{2}{x-2}\,\mathrm{d}x \quad = \quad \int \frac{-1}{x+5}\,\mathrm{d}x \; + \; \int \frac{2}{x-2}\,\mathrm{d}x$$
$$= \quad \boxed{-\ln|x+5| + 2\ln|x-2| + C.}$$

□

$(\frac{15}{200})$ (b) $\displaystyle\int \frac{\cos(x)^7}{\sin(x)^9}\,\mathrm{d}x.$

Solution: Recall that $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$, and $\cot'(x) = -\csc(x)^2 = \dfrac{-1}{\sin(x)^2}$. Thus,

$$\frac{\cos(x)^7}{\sin(x)^9} \quad = \quad \frac{\cos(x)^7}{\sin(x)^7}\cdot\frac{1}{\sin(x)^2} \quad = \quad \cot(x)^7\cdot\csc(x)^2 \quad = \quad -\cot(x)^7\cdot\cot'(x).$$

Thus,
$$\int \frac{\cos(x)^7}{\sin(x)^9}\,\mathrm{d}x \quad = \quad -\int \cot(x)^7\cdot\cot'(x)\,\mathrm{d}x \quad \underset{(*)}{=} \quad -\int u^7\,\mathrm{d}u \quad = \quad -\frac{u^8}{8} + C$$
$$\underset{(*)}{=} \quad \boxed{-\frac{\cot(x)^8}{8} + C.}$$

where $(*)$ is the substitution $u := \cot(x)$ so that $\mathrm{d}u = \cot'(x)\,\mathrm{d}x$.

Another solution: Recall that $\cos(x)^2 = 1 - \sin(x)^2$. Thus

$$\cos(x)^6 \quad = \quad (\cos(x)^2)^3 \quad = \quad (1-\sin(x)^2)^3 \quad = \quad 1 - 3\sin(x)^2 + 3\sin(x)^4 - \sin(x)^6.$$

Thus,
$$\frac{\cos(x)^7}{\sin(x)^9} \quad = \quad \frac{\cos(x)\cdot\cos(x)^6}{\sin(x)^9} \quad = \quad \frac{\left(1 - 3\sin(x)^2 + 3\sin(x)^4 - \sin(x)^6\right)\cdot\cos(x)}{\sin(x)^9}.$$

Thus,
$$\int \frac{\cos(x)^7}{\sin(x)^9}\,\mathrm{d}x \quad = \quad \int \frac{1 - 3\sin(x)^2 + 3\sin(x)^4 - \sin(x)^6}{\sin(x)^9}\cdot\cos(x)\,\mathrm{d}x$$
$$\underset{(*)}{=} \quad \int \frac{1 - 3u^2 + 3u^4 - u^6}{u^9}\,\mathrm{d}u \quad = \quad \int u^{-9} - 3u^{-7} + 3u^{-5} - u^{-3}\,\mathrm{d}u$$
$$= \quad \frac{-u^{-8}}{8} + \frac{3u^{-6}}{6} - \frac{3u^{-4}}{4} + \frac{u^{-2}}{2} + C$$
$$= \quad \frac{-1}{8\,u^8} + \frac{1}{2u^6} - \frac{3}{4\,u^4} + \frac{1}{2\,u^2} + C.$$
$$\underset{(*)}{=} \quad \boxed{\frac{-1}{8\,\sin(x)^8} + \frac{1}{2\sin(x)^6} - \frac{3}{4\,\sin(x)^4} + \frac{1}{2\,\sin(x)^2} \; + \; C.}$$

Here $(*)$ is the substitution $u := \sin(x)$, so that $\mathrm{d}u = \cos(x)\,\mathrm{d}x$. □

$(\frac{15}{200})$ (c) $\displaystyle\int x^5\,\ln(x)\,\mathrm{d}x.$

Solution: We use integration by parts. Let $u := \ln(x)$ and $\mathrm{d}v := x^5\,\mathrm{d}x$. Then $\mathrm{d}u = \dfrac{1}{x}\,\mathrm{d}x$ and $v = \dfrac{x^6}{6}$. Thus,

$$\int x^5\,\ln(x)\,\mathrm{d}x \quad = \quad \int u\,\mathrm{d}v \quad = \quad uv - \int v\,\mathrm{d}u \quad = \quad \frac{\ln(x)\,x^6}{6} - \int \frac{x^6}{6}\cdot\frac{1}{x}\,\mathrm{d}x$$

$$= \frac{\ln(x)\,x^6}{6} - \int \frac{x^5}{6}\,\mathrm{d}x \quad = \quad \boxed{\frac{\ln(x)\,x^6}{6} - \frac{x^6}{36} + C.}$$

$\square$

($\frac{10}{200}$) 4. Consider the improper integral $\displaystyle\int_1^\infty \frac{\arctan(x)\cdot\sqrt{x^3-1}}{x^5\cdot(1+\sin(x)^2)}\,\mathrm{d}x$. Is this integral convergent or divergent? (The antiderivative is not required.)

Solution: We will use the Comparison Test. Observe that $(1+\sin(x)^2) \geq 1$ and $0 \leq \arctan(x) \leq \frac{\pi}{2}$ for all $x > 0$. Thus,

$$\frac{\arctan(x)\cdot\sqrt{x^3-1}}{x^5\cdot(1+\sin(x)^2)} \quad \leq \quad \frac{\pi\cdot\sqrt{x^3-1}}{2x^5} \quad \leq \quad \frac{\pi\cdot\sqrt{x^3}}{2x^5} \quad = \quad \frac{\pi}{2}\frac{1}{x^{7/2}}$$

However, the improper integral $\displaystyle\int_1^\infty \frac{1}{x^{7/2}}\,\mathrm{d}x$ converges (because $7/2 > 1$). Thus, the Comparison Test says that the improper integral $\displaystyle\int_1^\infty \frac{\arctan(x)\cdot\sqrt{x^3-1}}{x^5\cdot(1+\sin(x)^2)}\,\mathrm{d}x$ also $\boxed{\text{converges.}}$ $\square$

5. Let $f(x) := 2x^3 + 3x^2 - 36x$.

($\frac{5}{200}$) (a) Find the intervals where f is increasing, and the intervals where f is decreasing.

Solution: $f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x+3)(x-2)$. Thus, f' has zeros at $x = -3$ and $x = 2$. We make a table to identify the increasing/decreasing intervals of f:

	$x+3$	$x-2$	$f'(x)$	f
$x < -3$	negative	negative	positive	increasing
$-3 < x < 2$	positive	negative	negative	decreasing
$2 < x$	positive	positive	positive	increasing

$\square$

($\frac{5}{200}$) (b) Find all the local maxima and local minima of f.

Solution: f is differentiable everywhere, so any extremal point of f must be a zero of f' (by Fermat's theorem). The only zeros of f' are at $x = -3$ and $x = 2$. According to the 'first derivative test', $\boxed{x = -3 \text{ is a local maximum}}$ and $\boxed{x = 2 \text{ is a local minimum.}}$

(The actual value at the maximum is $f(-3) = 81$. The value at the minimum is $f(2) = -44$. But you are not required to compute this.) $\square$

($\frac{5}{200}$) (c) Find the intervals where f is concave-up and the intervals where f is concave-down.

Solution: $f''(x) = 12x + 6 = 12(x + \frac{1}{2})$ Thus, $f''(x) = 0$ only if $x = \frac{-1}{2}$. We make a table to identify the concavity intervals of f:

	$12(x+\frac{1}{2})$	Concavity
$x < \frac{-1}{2}$	negative	concave down
$\frac{-1}{2} < x$	positive	concave up

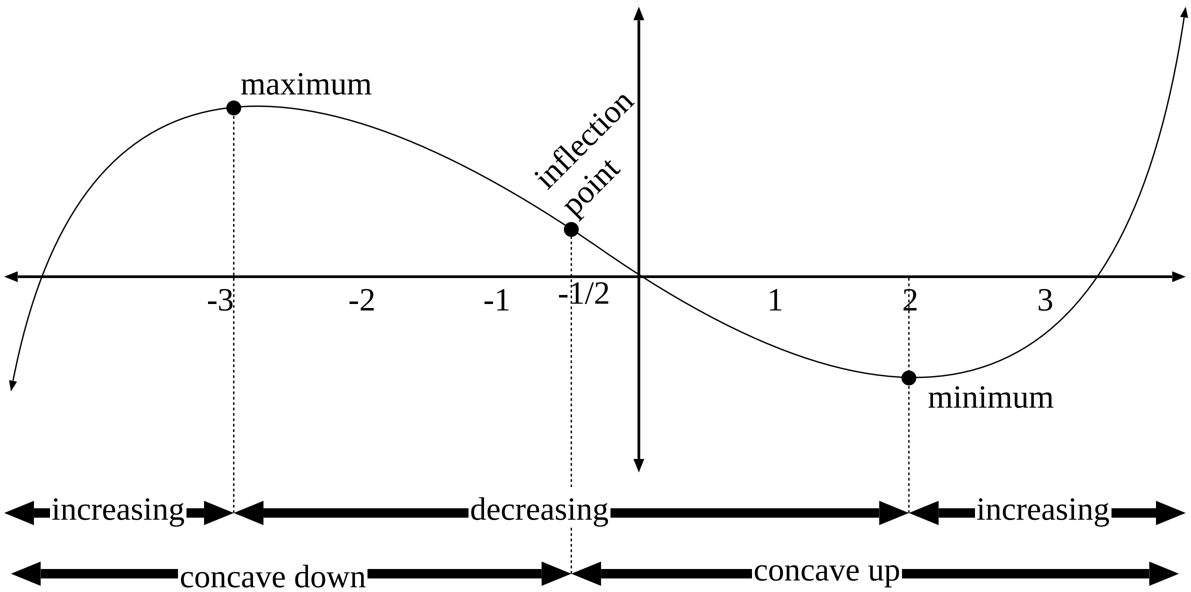
$\square$

($\frac{5}{200}$) (d) Find the inflection points of f.

Solution: The inflection points are places where f changes from concave up to concave down or vice versa. The only inflection point is at $x = \frac{-1}{2}$. $\square$

$(\frac{5}{200})$ (e) Use the information from parts (a)-(d) to sketch the graph of f. (Don't worry about x-intercepts).

Solution:



□

$(\frac{15}{200})$ 6. Compute $\lim_{x\to 0}(1-2x)^{1/x}$.

Solution: Note that $\lim_{x\to 0}(1-2x) = 1$ and $\lim_{x\to 0}(1/x) = \infty$. Thus, this is an indeterminate form of type "1^∞". Our first step is to take a logarithm:

$$\log\left(\lim_{x\to 0}(1-2x)^{1/x}\right) = \lim_{x\to 0}\log\left((1-2x)^{1/x}\right) = \lim_{x\to 0}\frac{\log(1-2x)}{x}. \tag{3}$$

Now, $\lim_{x\to 0}\log(1-2x) = \log(1) = 0$ and $\lim_{x\to 0} x = 0$, so we have an indeterminate form of type "$0/0$". If $f(x) = \log(1-2x)$ and $g(x) = x$, then $f'(x) = \dfrac{-2}{1-2x}$ and $g'(x) = 1$, so

$$\lim_{x\to 0}\frac{\log(1-2x)}{x} = \lim_{x\to 0}\frac{f(x)}{g(x)} \underset{\text{(H)}}{=} \lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{-2}{1-2x} = -2, \tag{4}$$

where (H) is by l'Hospital's rule. Combining (3) and (4), we get $\log\left(\lim_{x\to 0}(1-2x)^{1/x}\right) = -2$; thus, $\lim_{x\to 0}(1-2x)^{1/x} = \boxed{e^{-2}}$.

Common mistakes and remarks on grading: People got 5/15 for identifying this as an indeterminate form of type "0/0" and applying the natural logarithm (but then screwing up the rest of the problem).

Some people got all the way to the end, but then forgot to apply the exponential map and left "-2" as their final answer. This got 10/15 marks. □

$(\frac{15}{200})$ 7. Suppose $F(x) = \int_0^{x^2} \sqrt{1+r^3}\, dr$, for all $x \in \mathbb{R}$. Compute $F'(x)$.

Solution: Define the function $G : \mathbb{R}_+ \longrightarrow \mathbb{R}$ by $G(y) := \int_0^y \sqrt{1+r^3}\, dr$, for all $y \in \mathbb{R}_+$. Then $F(x) = G(x^2)$ for all $x \in \mathbb{R}$. Thus,

$$F'(x) \underset{(*)}{=} G'(x^2) \cdot 2x \underset{(\dagger)}{=} \boxed{\sqrt{1+x^6} \cdot 2x.}$$

Here, $(*)$ is the Chain rule, and $(\dagger)$ is because $G'(y) = \sqrt{1+y^3}$, by the Fundamental Theorem of Calculus.

Common mistakes and remarks on grading: A lot of people completely missed the point of this question. Instead of trying to use the Fundamental Theorem of Calculus, they attempted a brute force antidifferentiation. This, of course, is hopeless (and these people usually wrote a bunch of nonsense). This generally got around 3/15 marks.

People got 5/15 for indicating some vague understanding that the Fundamental Theorem of Calculus was relevant here (but failing to carry it through).

Some people basically had the right idea, but screwed up using the Chain Rule and ended up with the answer "$F'(x) = \sqrt{1+x^6}$". This got 10/15 marks.

One crazy person used Newton's Binomial Formula to get a MacLaurin series for the function $\sqrt{1+r^3}$, and then antidifferentiated this MacLaurin series. This is definitely way too complicated an approach. However, it was creative, and correctly used a high-powered technique to answer the problem, so this person got 15/15. □

8. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an unknown function, such that $\sin(f(x)) = \tan(x) \cdot \ln(x)$. Use this
$(\frac{10}{200})$ information to express $f'(x)$ as a function of x and $f(x)$.

Solution: We have

$$\begin{aligned}
& \sin(f(x)) &=& \tan(x) \cdot \ln(x). \\
\text{Differentiate to get:} \quad & \sin'(f(x)) \cdot f'(x) &=& \tan'(x) \cdot \ln(x) + \tan(x) \cdot \ln'(x). \\
\text{That is:} \quad & \cos(f(x)) \cdot f'(x) &=& \sec^2(x) \cdot \ln(x) + \frac{\tan(x)}{x}. \\
\text{Now simplify:} \quad & f'(x) &=& \boxed{\frac{1}{\cos(f(x))}\left(\sec^2(x) \cdot \ln(x) + \frac{\tan(x)}{x}\right).}
\end{aligned}$$

Common mistakes and remarks on grading: A lot of people wrote $f(x) = \arcsin(\tan(x) \cdot \ln(x))$ and then tried to differentiate this expression. Most of these people screwed this up because they couldn't apply the Chain Rule and/or Liebniz rule correctly. However, a few people did it properly and ended up with the answer:

$$f'(x) = \frac{1}{\sqrt{1 - \tan(x)^2 \cdot \ln(x)^2}}\left(\sec^2(x) \cdot \ln(x) + \frac{\tan(x)}{x}\right).$$

This answer is correct and got full marks. (By employing the hypothesized identity $\sin(f(x)) = \tan(x) \cdot \ln(x)$ and using Pythagoras' theorem, you can check that this is in fact equal to the previous answer.) □

9. Let $\mathcal{C}$ be the curve parameterized by $x(t) = 1 + 3t^2$ and $y(t) = 4 + t^3$, for $t \in [0, 1]$.

$(\frac{10}{200})$ (a) Find a formula for the slope $S(t) = \frac{dy}{dx}(t)$, as a function of t.

Solution: If $x(t) = 1 + 3t^2$, then $x'(t) = 6t$. If $y(t) = 4 + t^3$, then $y'(t) = 3t^2$. Thus, $S(t) = \frac{y'(t)}{x'(t)} = \frac{3t^2}{6t} = \boxed{\frac{t}{2}.}$ □

$(\frac{10}{200})$ (b) Compute the *arc-length* of $\mathcal{C}$.

Solution: If $x(t) = 1 + 3t^2$ and $y(t) = 4 + t^3$, then $x'(t) = 6t$ and $y'(t) = 3t^2$.

Thus, $x'(t)^2 = 36t^2$ and $y'(t)^2 = 9t^4$. Thus,

$$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{36t^2 + 9t^4} = \sqrt{9t^2(4 + t^2)}. = 3t\sqrt{t^2 + 4}.$$

$$\text{Thus,} \quad \text{arclength}(\mathcal{C}) = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^1 3t\sqrt{t^2 + 4}\, dt \underset{(*)}{=} \frac{3}{2}\int_4^5 \sqrt{u}\, du$$
$$= \frac{3}{2}\frac{2}{3}u^{3/2}\Big|_{u=4}^{u=5} = 5^{3/2} - 4^{3/2} = \boxed{\sqrt{125} - 8.}$$

Here, $(*)$ is the substitution $u := t^2 + 4$, so $du = 2t\, dt$, so $du/2 = t\, dt$. Also, $(t = 0) \Longrightarrow (u = 4)$ and $(t = 1) \Longrightarrow (u = 5)$.

Common mistakes and remarks on grading: Some people did a crazy trigonometric substitution $t := 2\tan(\theta)$ so that $dt = 2\sec(\theta)^2\, d\theta$ and then antidifferentiated the resulting expression to end up with the answer $8\sec^3(\arctan(1/2)) - 8$. By drawing the appropriate right-angle triangle you can check that in fact, $\sec^3(\arctan(1/2)) = \sqrt{125}$. So this answer is correct and got full marks.

□