Mathematics 110 – Calculus of one variable
Trent University 2003-2004

SOLUTIONS TO §A QUIZZES

Quiz #1. Friday, 19 September, 2002. *[10 minutes]*

12:00 Seminar

1. How close does x have to be to 1 in order to guarantee that $\frac{1}{x}$ is within $\frac{1}{10}$ of 1? *[10]*

SOLUTION. We'll reverse-engineer the "how close" (*i.e.* the δ) from the tolerance (*i.e.* the ε) we want to ensure.

$$\begin{aligned}
&\frac{1}{x} \text{ is within } \frac{1}{10} \text{ of } 1\\
\iff & -\frac{1}{10} < \frac{1}{x} - 1 < \frac{1}{10}\\
\iff & 1 - \frac{1}{10} < \frac{1}{x} < 1 + \frac{1}{10}\\
\iff & \frac{9}{10} < \frac{1}{x} < \frac{11}{10}\\
\iff & \frac{10}{9} > x > \frac{10}{11}\\
\iff & \frac{10}{9} - 1 > x - 1 > \frac{10}{11} - 1\\
\iff & \frac{1}{9} > x - 1 > -\frac{1}{11}\\
\Longleftarrow & \frac{1}{11} > x - 1 > -\frac{1}{11} \quad \text{since } \frac{1}{11} < \frac{1}{9}
\end{aligned}$$

Thus x ought to be within $\frac{1}{11}$ of 1 in order to guarantee that $\frac{1}{x}$ is within $\frac{1}{10}$ of 1. ■

13:00 Seminar

1. Find a value of $\delta > 0$ that ensures that $-1 < \sqrt{x} - 4 < 1$ whenever $-\delta < x - 16 < \delta$. *[10]*

SOLUTION. As usual, we reverse-engineer the δ from the ε, which in this case has the value 1.

$$\begin{aligned}
& -1 < \sqrt{x} - 4 < 1\\
\iff & 4 - 1 < \sqrt{x} < 4 + 1\\
\iff & 3 < sqrtx < 5\\
\iff & 9 < x < 25\\
\iff & 9 - 16 < x - 16 < 25 - 16\\
\iff & -7 < x - 16 < 9\\
\Longleftarrow & -7 < x - 16 < 9 \quad \text{since } 7 < 9
\end{aligned}$$

Thus $\delta = 7$ will ensure that $-1 < \sqrt{x} - 4 < 1$ whenever $-\delta < x - 16 < \delta$. ■

Leftovers

1. Use the $\varepsilon - \delta$ definition of limits to verify that that $\lim_{x \to 0} 1 = 1$.

Hint: Try any $\delta > 0$ you like ...

SOLUTION. There's not much to reverse-enginner here. Observe that for any $\varepsilon > 0$ whatsoever we have

$$-\varepsilon < 1 - 1 < \varepsilon$$

for the very simple reason that $1 - 1 = 0$. This is entirely indepependent of the value of x, so we get $-\varepsilon < 1 - 1 < \varepsilon$ whenever $-\delta < x - 0 < \delta$ no matter what value of $\delta > 0$ we try ... Hence, by the $\varepsilon - \delta$ definition of limit, $\lim_{x \to 0} 1 = 1$. ■

Quiz #2. Monday, 29 September, 2002. *[10 minutes]*

1. Use the $\varepsilon - \delta$ definition of limits to verify that that $\lim_{x \to 0} \sin^2(x) = 0$.

Hint: You may use the fact that $|\sin(x)| \leq |x|$.

SOLUTION. As usual, we reverse-engineer the δ from the ε.

$$\begin{aligned} & -\varepsilon < \sin^2(x) - 0 < \varepsilon \\ \iff & -\varepsilon < [\sin(x)]^2 < \varepsilon \\ \iff & -\sqrt{\varepsilon} < \sin(x) < \sqrt{\varepsilon} \\ \iff & |\sin(x)| < \sqrt{\varepsilon} \\ \Longleftarrow & |\sin(x)| \leq |x| < \sqrt{\varepsilon} \\ \Longleftarrow & |x - 0| < \sqrt{\varepsilon} \end{aligned}$$

It follows that $\delta = \sqrt{\varepsilon}$ will do the job, and hence that $\lim_{x \to 0} \sin^2(x) = 0$. ■

Quiz #3. Friday, 3 October, 2003. *[15 minutes]*

12:00 Seminar

Evaluate

$$1. \quad \lim_{x \to \infty} \frac{x - 2}{x^2 - 3x + 2} \quad [5] \qquad 2. \quad \lim_{x \to 0} \frac{e^x - 1}{e^{2x} - 1} \quad [5]$$

SOLUTIONS.

1.
$$\lim_{x \to \infty} \frac{x - 2}{x^2 - 3x + 2} = \lim_{x \to \infty} \frac{x - 2}{(x - 2)(x - 1)} = \lim_{x \to \infty} \frac{1}{x - 1} = 0$$

since $\lim_{x \to \infty} (x - 1) = \infty$. ■

2.
$$\lim_{x \to 0} \frac{e^x - 1}{e^{2x} - 1} = \lim_{x \to 0} \frac{e^x - 1}{(e^x)^2 - 1} = \lim_{x \to 0} \frac{e^x - 1}{(e^x - 1)(e^x + 1)} = \lim_{x \to 0} \frac{1}{e^x + 1} = \frac{1}{1 + 1} = \frac{1}{2}$$

since $\lim_{x \to 0} e^x = e^0 = 1$. ■

13:00 Seminar

Evaluate

$$1.\quad \lim_{x\to 1} \frac{x^2+x-2}{x^2+2x-3} \quad \textit{[5]} \qquad 2.\quad \lim_{x\to\infty} \frac{(x+4)^2}{41x^2+43x+47} \quad \textit{[5]}$$

SOLUTIONS.

1.

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2+2x-3} = \lim_{x\to 1} \frac{(x-1)(x+2)}{(x-1)(x+3)} = \lim_{x\to 1} \frac{x+2}{x+3} = \frac{1+2}{1+3} = \frac{3}{4} \quad \blacksquare$$

2.

$$\begin{aligned}\lim_{x\to\infty} \frac{(x+4)^2}{41x^2+43x+47} &= \lim_{x\to\infty} \frac{x^2+8x+16}{41x^2+43x+47} = \lim_{x\to\infty} \frac{x^2+8x+16}{41x^2+43x+47} \cdot \frac{1/x^2}{1/x^2} \\ &= \lim_{x\to\infty} \frac{1+\frac{8}{x}+\frac{16}{x^2}}{41+\frac{43}{x}+\frac{47}{x^2}} = \frac{1+0+0}{1+0+0} = 1\end{aligned}$$

since $\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to\infty} \frac{1}{x^2} = 0$. ■

Leftovers

1. For what value(s) of the constant c does $\lim_{x\to 2}(cx+3) = \lim_{t\to\infty} \frac{ct^2+3+c}{t^2+1}$? *[10]* SOLUTION. First, we'll evaluate both limits:

$$\lim_{x\to 2}(cx+3) = c\cdot 2+3 = 2c+3$$

and

$$\lim_{t\to\infty} \frac{ct^2+3+c}{t^2+1} = \lim_{t\to\infty} \frac{ct^2+3+c}{t^2+1} \cdot \frac{1/t^2}{1/t^2} = \lim_{t\to\infty} \frac{c+\frac{3}{t}+\frac{c}{t^2}}{1+\frac{1}{t^2}} = \frac{c+0+0}{1+0} = c$$

Second, setting them equal gives the linear equation $2c+3=c$; solving this equation for c gives $c=-3$. ■

Quiz #4. Friday, 10 October, 2003. *[10 minutes]*

12:00 Seminar

1. Use the limit definition of the derivative to find $f'(0)$ if $f(x) = (x+1)^3$. *[10]*

SOLUTION.

$$\begin{aligned}f'(0) &= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{(0+h+1)^2-(0+1)^2}{h} = \lim_{h\to 0} \frac{(h+1)^2-1}{h} \\ &= \lim_{h\to 0} \frac{h^2+2h+1-1}{h} = \lim_{h\to 0} \frac{h^2+2h}{h} = \lim_{h\to 0}(h+2) = 0+2 = 2 \quad \blacksquare\end{aligned}$$

13:00 Seminar

1. Use the limit definition of the derivative to find $f'(x)$ if $f(x) = \dfrac{1}{x}$. *[10]*

SOLUTION.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h}$$
$$= \lim_{h\to 0} \frac{-h}{hx(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x(x+0)} = \frac{-1}{x^2} \quad \blacksquare$$

Leftovers

1. Use the limit definition of the derivative to find $f'(x)$ if $f(x) = x^2 - 3x$. *[10]*

SOLUTION.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left[(x+h)^2 - 3(x+h)\right] - \left[x^2-3x\right]}{h}$$
$$= \lim_{h\to 0} \frac{\left[x^2+2xh+h^2-3x-3h\right] - \left[x^2-3x\right]}{h}$$
$$= \lim_{h\to 0} \frac{x^2+2xh+h^2-3x-3h-x^2+3x}{h}$$
$$= \lim_{h\to 0} \frac{2xh+h^2-3h}{h} = \lim_{h\to 0} (2x+h-3) = 2x+0-3 = 2x-3 \quad \blacksquare$$

Quiz #5. Friday, 17 October, 2003. *[10 minutes]*

12:00 Seminar

Find $\frac{dy}{dx}$ in each of the following:

1. $y = \ln(\sec(x) + \tan(x))$ *[3]* 2. $e^{xy} = 2$ *[3]* 3. $y = \dfrac{x^2+4x+4}{x+3}$ *[4]*

SOLUTIONS.

1.

$$\frac{dy}{dx} = \frac{d}{dx}\ln(\sec(x)+\tan(x))$$
$$= \frac{1}{\sec(x)+\tan(x)} \cdot \frac{d}{dx}(\sec(x)+\tan(x))$$
$$= \frac{1}{\sec(x)+\tan(x)} \cdot \left(\frac{d}{dx}\sec(x) + \frac{d}{dx}\tan(x)\right)$$
$$= \frac{1}{\sec(x)+\tan(x)} \cdot \left(\sec(x)\tan(x) + \sec^2(x)\right)$$
$$= \frac{\sec(x)(\tan(x)+\sec(x))}{\sec(x)+\tan(x)} = \sec(x) \quad \blacksquare$$

2.

$$e^{xy} = 2 \iff xy = \ln(2) \iff y = \frac{\ln(2)}{x}$$

so

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ln(2)}{x}\right) = \ln(2)\frac{d}{dx}\left(\frac{1}{x}\right) = \ln(2)\cdot\frac{-1}{x^2} = -\frac{\ln(2)}{x^2} \quad \blacksquare$$

3.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x^2+4x+4}{x+3}\right) \\
&= \frac{\frac{d}{dx}\left(x^2+4x+4\right)\cdot(x+3) - \left(x^2+4x+4\right)\cdot\frac{d}{dx}(x+3)}{(x+3)^2} \\
&= \frac{(2x+4)(x+3) - \left(x^2+4x+4\right)\cdot 1}{(x+3)^2} \\
&= \frac{2x^2+10x+12-x^2-4x-4}{(x+3)^2} \\
&= \frac{x^2+6x+8}{(x+3)^2} = \frac{(x+2)(x+4)}{(x+3)^2} \quad \blacksquare
\end{aligned}$$

13:00 Seminar

Find $\frac{dy}{dx}$ in each of the following:

1. $y = \left(1+x^2\right)\arctan(x)$ *[3]* 2. $\tan(x+y) = 1$ *[3]* 3. $y = \dfrac{e^x+1}{e^{2x}-1}$ *[4]*

SOLUTIONS.

1.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left[\left(1+x^2\right)\arctan(x)\right] = \frac{d}{dx}\left(1+x^2\right)\cdot\arctan(x) + \left(1+x^2\right)\cdot\frac{d}{dx}\arctan(x) \\
&= 2x\arctan(x) + \left(1+x^2\right)\cdot\frac{1}{1+x^2} = 2x\arctan(x)+1 \quad \blacksquare
\end{aligned}$$

2. The hard way:

$$\begin{aligned}
&\tan(x+y) = 1 \\
\implies &\frac{d}{dx}\tan(x+y) = \frac{d}{dx}1 \\
\implies &\sec^2(x+y)\cdot\frac{d}{dx}(x+y) = 0 \\
\implies &\sec^2(x+y)\cdot\left(\frac{d}{dx}x + \frac{d}{dx}y\right) = 0 \\
\implies &\sec^2(x+y)\cdot\left(1+\frac{dy}{dx}\right) = 0 \\
\implies &1+\frac{dy}{dx} = 0 \qquad \text{(since } \sec(t) \neq 0 \text{ when defined)} \\
\implies &\frac{dy}{dx} = -1
\end{aligned}$$

... and the easy way:

$$\begin{aligned} \tan(x+y) = 1 &\iff x + y = n\pi + \frac{\pi}{4} \quad \text{for some integer } n \\ &\iff y = -x + n\pi + \frac{\pi}{4} \quad \text{for some integer } n \end{aligned}$$

so

$$\frac{dy}{dx} = \frac{d}{dx}\left(-x + n\pi + \frac{\pi}{4}\right) = -1 + 0 = -1 \quad \blacksquare$$

3.

$$y = \frac{e^x + 1}{e^{2x} - 1} = \frac{e^x + 1}{(e^x)^2 - 1} = \frac{e^x + 1}{(e^x - 1)(e^x + 1)} = \frac{1}{e^x - 1}$$

so

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{e^x - 1}\right) = \frac{-1}{(e^x - 1)^2} \cdot \frac{d}{dx}(e^x - 1) = \frac{-1}{(e^x - 1)^2} \cdot e^x = \frac{-e^x}{(e^x - 1)^2} \quad \blacksquare$$

Leftovers

Find $\frac{dy}{dx}$ in each of the following:

1. $y = \sqrt{1 - e^{2x}}$ *[3]* 2. $y = \dfrac{\tan(x)}{\cos(x)}$ *[3]* 3. $\ln(x + y) = x$ *[4]*

SOLUTIONS.

1.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\sqrt{1 - e^{2x}} = \frac{d}{dx}\left(1 - e^{2x}\right)^{1/2} = \frac{1}{2}\left(1 - e^{2x}\right)^{-1/2} \cdot \frac{d}{dx}\left(1 - e^{2x}\right) \\ &= \frac{1}{2}\left(1 - e^{2x}\right)^{-1/2} \cdot \left(0 - e^{2x} \cdot \frac{d}{dx}(2x)\right) = \frac{1}{2}\left(1 - e^{2x}\right)^{-1/2} \cdot \left(-2e^{2x}\right) \\ &= -e^{2x}\left(1 - e^{2x}\right)^{-1/2} = \frac{-e^{2x}}{\sqrt{1 - e^{2x}}} \quad \blacksquare \end{aligned}$$

2.

$$y = \frac{\tan(x)}{\cos(x)} = \tan(x)\frac{1}{\cos(x)} = \tan(x)\sec(x)$$

so

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\tan(x)\sec(x)\right) = \left(\frac{d}{dx}\tan(x)\right) \cdot \sec(x) + \tan(x) \cdot \left(\frac{d}{dx}\sec(x)\right) \\ &= \sec^2(x)\sec(x) + \tan(x)\sec(x)\tan(x) = \sec(x)\left(\sec^2(x) + \tan^2(x)\right) \quad \blacksquare \end{aligned}$$

3. The hard way:

$$\begin{aligned}
\ln(x+y) = x &\Longrightarrow \frac{d}{dx}\ln(x+y) = \frac{d}{dx}x \\
&\Longrightarrow \frac{1}{x+y}\cdot\frac{d}{dx}(x+y) = 1 \\
&\Longrightarrow \frac{1}{x+y}\cdot\left(1+\frac{dy}{dx}\right) = 1 \\
&\Longrightarrow 1+\frac{dy}{dx} = x+y \\
&\Longrightarrow \frac{dy}{dx} = x+y-1
\end{aligned}$$

… and the easy way:

$$\ln(x+y) = x \iff x+y = e^x \iff y = e^x - x$$

so

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x - x\right) = e^x - 1 \quad \blacksquare$$