Mathematics 110 – Calculus of one variable
§A FINAL EXAMINATION
Trent University, 8 April, 2004

Time: 3 hours *Brought to you by* Стефан Біланюк.

Instructions: Show all your work and justify all your answers. *If in doubt,* **ask!**

Aids: Calculator; an $8.5'' \times 11''$ aid sheet or the pamphlet *Formula for Success*; one brain.

Part I. Do all three of **1** – **4**.

1. Find $\frac{dy}{dx}$ (in terms of x and/or y) in any *three* of **a** – **f**. *[15 = 3 × 5 ea.]*

a. $y = x^2 \sin(x+3)$ **b.** $e^{xy} = 2$ **c.** $y = \cos^2\left(x^2\right)$

d. $\begin{array}{l} y = t^2 \\ x = t^3 \end{array}$ **e.** $y = \int_0^{x^2} \sqrt{w}\, dw$ **f.** $y = \frac{\cos(x)}{1+\tan(x)}$

2. Evaluate any *three* of the integrals **a** – **f**. *[15 = 3 × 5 ea.]*

a. $\int_e^\infty \frac{1}{x\ln(x)}\, dx$ **b.** $\int xe^{-x}\, dx$ **c.** $\int_{-1}^1 \frac{2s}{1+s^4}\, ds$

d. $\int \frac{1}{\sqrt{x^2+4}}\, dx$ **e.** $\int_0^1 \arctan(t)\, dt$ **f.** $\int \frac{3x-3}{x^2+x-2}\, dx$

3. Determine whether the series converges absolutely, converges conditionally, or diverges in any *two* of **a** – **d**. *[10 = 2 × 5 ea.]*

a. $\sum_{n=1}^\infty \frac{(-1)^n}{n^2}$ **b.** $\sum_{n=0}^\infty \left(3^n - 2^n\right)$ **c.** $\sum_{n=0}^\infty \frac{n}{1+n^2}$ **d.** $\sum_{n=0}^\infty 3^{-n}2^n\cos(n\pi)$

4. Do any *three* of **a** – **f**. *[15 = 3 × 5 ea.]*

a. Use an $\varepsilon - N$ argument to verify that $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

b. Sketch the polar curve $r = \theta$ for $-\pi \le \theta \le \pi$ and find its slope at $\theta = 0$.

c. Evaluate $\lim_{x\to 0} \frac{x^2}{\tan(x)}$ or show that the limit does not exist.

d. Use the Right-hand Rule to compute the definite integral $\int_0^2 (x+1)\, dx$.

e. Find the area of the surface obtained by rotating the curve $y = 2x$, $0 \le x \le 2$, about the y-axis.

f. Determine whether $f(x) = \begin{cases} 1 - e^x & x \le 0 \\ \ln(x+1) & x > 0 \end{cases}$ is continuous at $x = 0$ or not.

Part II. Do *one* of **5** or **6**.

5. A container of volume $54\pi\ cm^3$ is made from sheet metal. Find the dimensions of such a container which require the least amount sheet metal to make if:

a. The container is cylindrical, with a bottom but without a top. *[13]*

b. The container is a sphere. *[2]*

Hint: The volume of a cylinder of radius r and height h is $\pi r^2 h$; the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

6. Find the domain, all maximum, minimum, and inflection points, and all vertical and horizontal asymptotes of $f(x) = \dfrac{x}{1+x^2}$, and sketch its graph. *[15]*

Part III. Do *one* of **7** or **8**.

7. Find the surface area of a cone with height 8 cm and radius 2 cm at the base. *[15]*

8. Consider the region bounded above by $y = \sin(x)$ and below by $y = \dfrac{2}{\pi}x$ for $0 \leq x \leq \dfrac{\pi}{2}$.

a. Sketch this region. *[2]*

b. Sketch the solid obtained by revolving this region about the x-axis. *[3]*

c. Find the volume of this solid. *[10]*

Part IV. Do *one* of **9** or **10**.

9. Let $f(x) = \cos(x)$.

a. Find the Taylor series of $f(x)$ centred at $a = \pi$. *[8]*

b. Determine the radius and interval of convergence of this Taylor series. *[4]*

c. Use your work for **a** to help find the Taylor series of $g(x) = \sin(x)$ at $a = \pi$. *[3]*

10. Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$.

a. Find the radius and interval of convergence of this power series. *[7]*

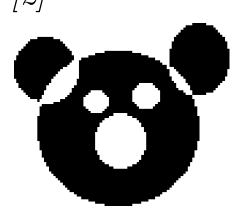
b. What function has this power series as its Taylor series at $a = 0$? *[8]*

[Total = 100]

Part i. Bonus!

$s\pi$**.** Write a little poem about calculus or mathematics in general. *[2]*

$+\iota$**.** Suppose a number of circles are drawn on a piece of paper, dividing it up into regions whose borders are made up of circular arcs. Prove that you can colour these regions with only *two* colours in such a way that no two regions that have a common border have the same colour. *[2]*



I HOPE THAT YOU ENJOYED THE COURSE! ENJOY THE SUMMER TOO!