Mathematics 110 – Calculus of one variable

TRENT UNIVERSITY 2003-2004

SOLUTION TO ASSIGNMENT #5

Suppose the ends of a string are attached to two fixed points and it hangs under its own weight. (To keep things relatively simple, we'll assume the string is completely flexible, of uniform composition and density, arbitrarily strong, and undisturbed by any forces aside from gravity and the attachment at those two fixed points.) It turns out that the curve $y = f(x)$ formed by such a string must satisfy a differential equation of the form

$$\frac{d^2y}{dx^2} = k\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

where $k > 0$ is a constant. (See Chapter 9 of the text for basic information about differential equations. For the reasons why the curve must satisfy such a differential equation, take some physics ...)

1. Solve the differential equation for $y = f(x)$ if the two fixed points between which the string is suspended are at $(-2, 2)$ and $(2, 2)$ and the lowest point of the string is at $(0, 1)$. *[10]*

Hint: Let $z = \frac{dy}{dx}$. Rewrite the differential equation in terms of z. Solve the new equation for z. Integrate z to get y. Finally, use the points that you know $y = f(x)$ passes through to solve for the constants that appear in y. You may wish to check out §3.9 in the text for information about the hyperbolic functions, which may play a role in this problem.

SOLUTION. Following the hint, we plug $z = \dfrac{dy}{dx}$ into the equation

$$\frac{d^2y}{dx^2} = k\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

to get

$$\frac{dz}{dx} = k\sqrt{1 + z^2}\,.$$

Note that $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{dz}{dx}$.

The equation in z is a *separable* differential equation (see §9.3 of the text for more information about these), which are so-called because they can be rearranged to put everything involving each of the two variables on different sides of the equation:

$$\begin{aligned} \frac{dz}{dx} &= k\sqrt{1 + z^2} \\ \Rightarrow \quad dz &= k\sqrt{1 + z^2}\, dx \\ \Rightarrow \quad \frac{1}{\sqrt{1 + z^2}}\, dz &= k dx \end{aligned}$$

We can now integrate both sides of the last equation, each with respect to the variable involved:

$$\int \frac{1}{\sqrt{1+z^2}}\,dz = \int k\,dx$$

The right-hand side is very easy to do:

$$\int k\,dx = kx + \text{a constant}$$

The left-hand side you can do with a trig substitution or just look it up in the tables in the back of the text:

$$\int \frac{1}{\sqrt{1+z^2}}\,dz = \ln\left(z + \sqrt{1+z^2}\right) + \text{another constant}$$

It follows that

$$\ln\left(z + \sqrt{1+z^2}\right) + \text{another constant} = kx + \text{a constant},$$

so

$$\ln\left(z + \sqrt{1+z^2}\right) = kx + C\,,$$

where $C = \text{a constant} - \text{another constant}$.

We next have to solve the equation above for z as a function of x. The first step, getting rid of the logarithm by throwing the exponential function at both sides, is quick:

$$z + \sqrt{1+z^2} = e^{kx+C}$$

Isolating z now requires some algebra. Here are the highlights:

$$\begin{aligned}
& z + \sqrt{1+z^2} = e^{kx+C} \\
\Rightarrow\quad & \sqrt{1+z^2} = e^{kx+C} - z \\
\Rightarrow\quad & 1 + z^2 = \left(e^{kx+C} - z\right)^2 = e^{2(kx+C)} - 2e^{kx+C}z + z^2 \\
\Rightarrow\quad & 1 = e^{2(kx+C)} - 2e^{kx+C}z \\
\Rightarrow\quad & z = \frac{e^{2(kx+C)} - 1}{2e^{kx+C}}
\end{aligned}$$

A little rearranging puts this in a more convenient form:

$$z = \frac{e^{kx+C} - e^{-(kx+C)}}{2} = \sinh(kx+C)$$

For more about sinh and the other hyperbolic functions, see §3.9 of the text.

Recall that $z = \frac{dy}{dx}$. We can now get y as a function of x by integrating $z = \frac{dy}{dx}$:

$$\begin{aligned} y &= \int \frac{dy}{dx}\,dx = \int \frac{e^{kx+C} - e^{-(kx+C)}}{2}\,dx \\ &= \frac{1}{k}\cdot\frac{e^{kx+C} + e^{-(kx+C)}}{2} + B = \frac{1}{k}\cosh(kx+C) + B \end{aligned}$$

The details of the integration we leave to you. Note that another constant, namely B, turns up at this stage.

One task remains: to determine the unknown constants k, C, and B. Note that $k \neq 0$. (If it were otherwise, then $\frac{d^2y}{dx^2} = 0$, so the graph of y would be a straight line, which is impossible because the three points it is supposed to pass through form a triangle.) What we have going for us is the fact that the function passes through the points $(-2, 2)$, $(2, 2)$, and $(0, 1)$. This boils down to:

$$\begin{aligned} \frac{1}{k}\cosh(-2k+C) + B &= 2 \\ \frac{1}{k}\cosh(2k+C) + B &= 2 \\ \frac{1}{k}\cosh(C) + B &= 1 \end{aligned}$$

It follows from the first two of these equations that $\cosh(-2k+C) = \cosh(2k+C)$, so:

$$\begin{aligned} & \frac{e^{-2k+C} + e^{-(-2k+C)}}{2} = \frac{e^{2k+C} + e^{-(2k+C)}}{2} \\ \Rightarrow\quad & e^{-2k+C} + e^{2k-C} = e^{2k+C} + e^{-2k-C} \\ \Rightarrow\quad & e^{C}e^{-2k} + e^{-C}e^{2k} = e^{C}e^{2k} + e^{-C}e^{-2k} \\ \Rightarrow\quad & e^{-C}e^{2k} - e^{-C}e^{-2k} = e^{C}e^{2k} - e^{C}e^{-2k} \\ \Rightarrow\quad & e^{-C}\left(e^{2k} - e^{-2k}\right) = e^{C}\left(e^{2k} - e^{-2k}\right) \\ \Rightarrow\quad & e^{-C} = e^{C} \quad \textbf{or} \quad e^{2k} - e^{-2k} = 0 \\ \Rightarrow\quad & -C = C \quad \textbf{or} \quad e^{2k} = e^{-2k} \\ \Rightarrow\quad & C = 0 \quad \textbf{or} \quad 2k = -2k \\ \Rightarrow\quad & C = 0 \quad \textbf{or} \quad k = 0 \end{aligned}$$

We have already noted that $k \neq 0$, so it must be the case that $C = 0$.

Because $\cosh(t) = \cosh(-t)$ for any t and $C = 0$,

$$\cosh(-2k+C) = \cosh(-2k) = \cosh(2k) = \cosh(2k+C)\,.$$

The three equations obtained by plugging in the three points now boil down to two equations involving the constants k and B:

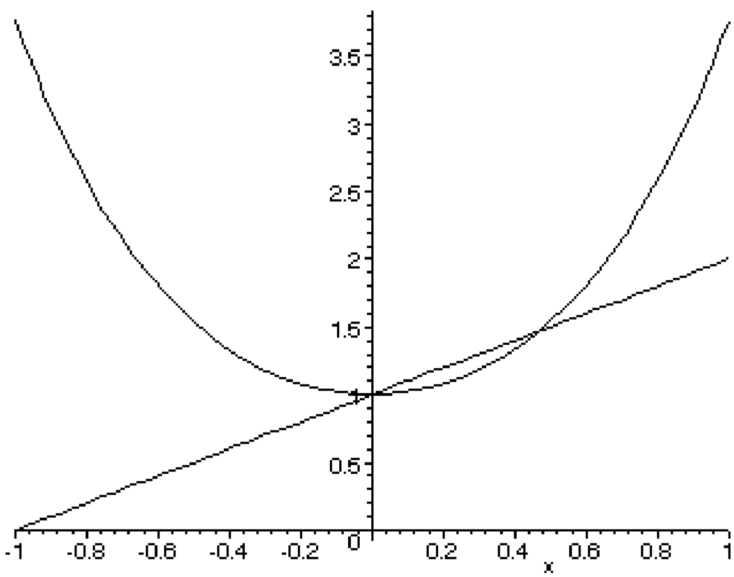
$$\begin{aligned} \frac{1}{k}\cosh(2k) + B &= 2 \\ \frac{1}{k} + B &= 1 \end{aligned}$$

(Note that $\cosh(0) = \dfrac{e^0 + e^{-0}}{2} = \dfrac{1+1}{2} = 1$.) The second of these equations implies that $B = 1 - \dfrac{1}{k} = \dfrac{k-1}{k}$ and $k = \dfrac{1}{1-B}$. Plugging $B = 1 - \dfrac{1}{k}$ into the first equation gives:

$$\begin{aligned} & \frac{1}{k}\cosh(2k) + 1 - \frac{1}{k} = 2 \\ \Rightarrow\quad & \frac{1}{k}\left(\cosh(2k) - 1\right) = 1 \\ \Rightarrow\quad & \cosh(2k) - 1 = k \\ \Rightarrow\quad & \cosh(2k) = k + 1 \end{aligned}$$

$k = 0$ would be one solution to this equation, but we already know that $k \neq 0$. To see that there must be another solution, consider the graph below, generated in MAPLE with the command:

```
plot( [cosh(2*x), x+1], x=-1..1, color=[black,black] );
```



From the graph, it would appear that the other point of intersection of the graphs of $y = \cosh(2x)$ and $y = x + 1$ is at approximately $x = 0.5$. The MAPLE command

```
fsolve( cosh(2*x) = x+1, x, 0.25..0.75 );
```

tells us that the point is approximately (still!) 0.4654105968. Thus

$$k \approx 0.4654105968\,, \quad \frac{1}{k} \approx 2.148640377\,, \quad \text{and} \quad B = 1 - \frac{1}{k} \approx -1.148640377\,,$$

so the equation of the curve formed by the string is approximately:

$$y \approx 2.148640377 \cdot \cosh(0.4654105968 \cdot x) - 1.148640377$$

Just for mathochistic fun, try to work out algebraically exactly what k is from $\cosh(2k) = k + 1$. ■