Mathematics 110 – Calculus of one variable

Trent University 2003-2004

SOLUTIONS TO ASSIGNMENT #4

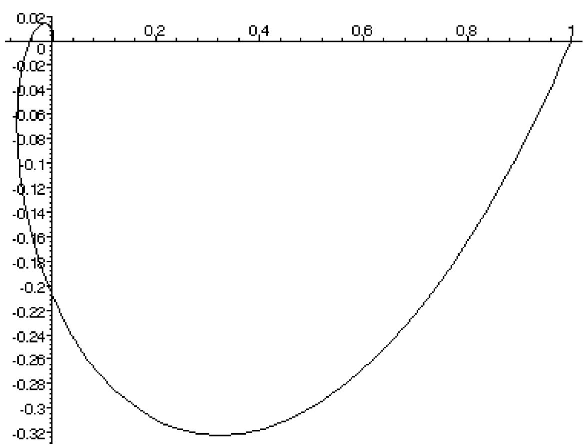
Consider the curve given by the following parametric equations.

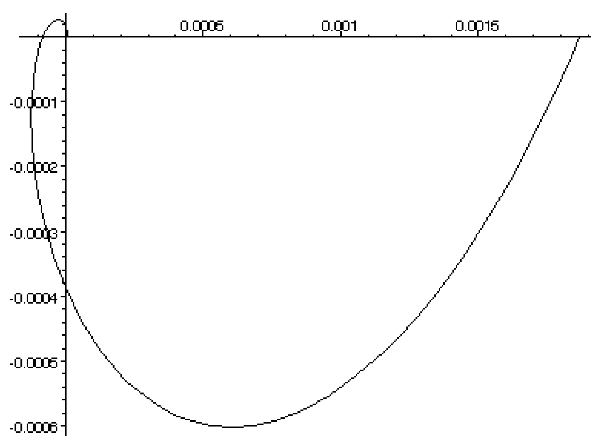
$$x = e^t \cos(t)$$
$$y = e^t \sin(t)$$
$$\text{where} \quad -\infty < t \leq 0$$

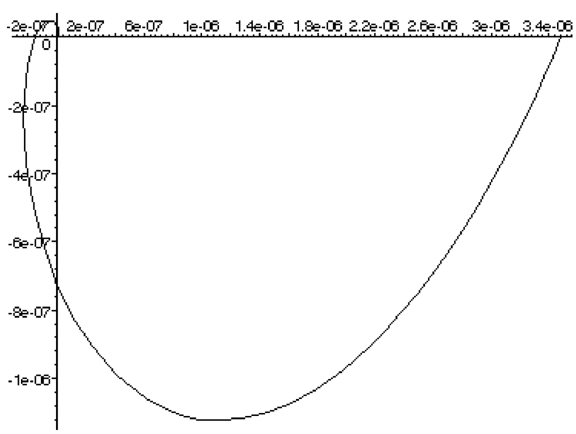
(See §10.1 and 10.2 in the text for information on how to handle curves in parametric form.)

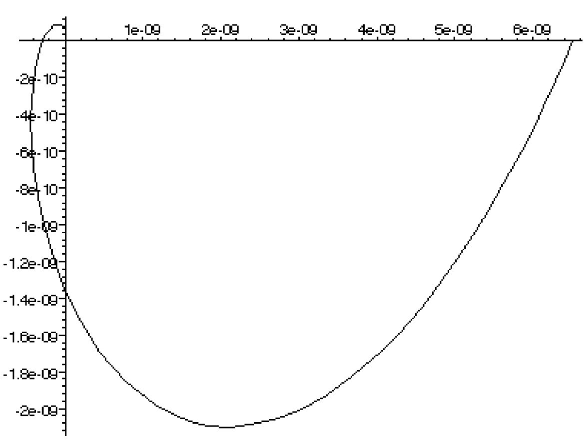
1. Sketch this curve. *[2]*

SOLUTION. Here are graphs of the curve on the four intervals $[-2\pi, 0]$, $[-4\pi, -2\pi]$, $[-6\pi, -4\pi]$, and $[-8\pi, -6\pi]$, respectively. Note the changes in scale in the graphs . . .









The curve is actually a spiral, but it's a little hard to tell that when graphing it, since it approaches approaches the origin so quickly. This happens because $e^t \to 0$ *very* quickly as $t \to -\infty$.

The graphs were generated in `Maple` using the commands

```
plot( [exp(t)*cos(t), exp(t)*sin(t), t=-2*Pi..0] );
plot( [exp(t)*cos(t), exp(t)*sin(t), t=-4*Pi..-2*Pi] );
plot( [exp(t)*cos(t), exp(t)*sin(t), t=-6*Pi..-4*Pi] );
plot( [exp(t)*cos(t), exp(t)*sin(t), t=-8*Pi..-6*Pi] );
```

respectively. ■

2. Find the length of this curve. *[4]*

SOLUTION. We throw the parametric version of the arc-length formula at this curve. First, note that

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}\left(e^t\cos(t)\right) = \left(\frac{d}{dt}e^t\right)\cdot\cos(t) + e^t\cdot\left(\frac{d}{dt}\cos(t)\right)\\ &= e^t\cos(t) + e^t\left(-\sin(t)\right) = e^t\left(\cos(t)-\sin(t)\right)\end{aligned}$$

and

$$\begin{aligned}\frac{dy}{dt} &= \frac{d}{dt}\left(e^t\sin(t)\right) = \left(\frac{d}{dt}e^t\right)\cdot\sin(t) + e^t\cdot\left(\frac{d}{dt}\sin(t)\right)\\ &= e^t\sin(t) + e^t\left(\cos(t)\right) = e^t\left(\sin(t)+\cos(t)\right)\,.\end{aligned}$$

Now we plug in and go:

$$\begin{aligned}\int_C ds &= \int_{-\infty}^0 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt\\
&= \int_{-\infty}^0 \sqrt{\left(e^t\left(\cos(t)-\sin(t)\right)\right)^2 + \left(e^t\left(\sin(t)+\cos(t)\right)\right)^2}\,dt\\
&= \int_{-\infty}^0 \sqrt{e^{2t}\left(\cos(t)-\sin(t)\right)^2 + e^{2t}\left(\sin(t)+\cos(t)\right)^2}\,dt\\
&= \int_{-\infty}^0 \sqrt{e^{2t}\left(\cos^2(t) - 2\cos(t)\sin(t) + \sin^2(t) + \sin^2(t) + 2\sin(t)\cos(t) + \cos^2(t)\right)}\,dt\\
&= \int_{-\infty}^0 \sqrt{e^{2t}\left(2\cos^2(t) + 2\sin^2(t)\right)}\,dt = \int_{-\infty}^0 \sqrt{2e^{2t}\left(\cos^2(t)+\sin^2(t)\right)}\,dt\\
&= \int_{-\infty}^0 \sqrt{2e^{2t}(1)}\,dt = \int_{-\infty}^0 \sqrt{2}\sqrt{e^{2t}}\,dt = \sqrt{2}\int_{-\infty}^0 e^t\,dt = \sqrt{2}\lim_{u\to-\infty}\int_u^0 e^t\,dt\\
&= \sqrt{2}\lim_{u\to-\infty} e^t\Big|_u^0 = \sqrt{2}\lim_{u\to-\infty}\left(e^0 - e^u\right) = \sqrt{2}\lim_{u\to-\infty}\left(1-e^u\right) = \sqrt{2}\,(1-0)\\
&\qquad\text{because } e^u \to 0 \text{ as } u \to -\infty\\
&= \sqrt{2}\end{aligned}$$

■

3. Suppose the curve is rotated about the x-axis. What is the area of the resulting surface? *[4]*

SOLUTION. The formula for the area of the surface obtained by rotating a curve about a line is $\int_C 2\pi r\, ds$. We know what the limits and ds are for this curve from problem **2**, but we need to figure out what r is. Since we are rotating the curve about the x-axis, r will be the distance from a point on the curve to the x-axis. The problem is that the curve is sometimes above the x-axis and sometimes below it as it spirals around the origin, so we can't just use $r = y - 0 = y$. Instead, we have to use $r = |y - 0| = |y|$:

$$\begin{aligned}
\int_C 2\pi r\, ds &= \int_{-\infty}^{0} 2\pi |y| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_{-\infty}^{0} 2\pi \left|e^t \sin(t)\right| \sqrt{2} e^t\, dt \\
&= 2\sqrt{2}\pi \int_{-\infty}^{0} e^t \left|\sin(t)\right| e^t\, dt \qquad \text{since } e^t > 0 \text{ for all } t \\
&= 2\sqrt{2}\pi \int_{-\infty}^{0} e^{2t} \left|\sin(t)\right|\, dt
\end{aligned}$$

Our problem now is that we have to break the integral up into pieces according to where $\sin(t)$ is positive or negative: when $\sin(t) > 0$, we have $|\sin(t)| = \sin(t)$, and when $\sin(t) < 0$, we have $|\sin(t)| = -\sin(t)$. $\sin(t)$ is negative on $[-\pi, 0]$, positive on $[-2\pi, -\pi]$, negative on $[-3\pi, -2\pi]$, positive on $[-4\pi, -3\pi]$, and so on. It follows that:

$$\begin{aligned}
\int_C 2\pi r\, ds &= 2\sqrt{2}\pi \int_{-\infty}^{0} e^{2t} \left|\sin(t)\right|\, dt \\
&= 2\sqrt{2}\pi \left[\int_{-\pi}^{0} e^{2t} \left(-\sin(t)\right) dt + \int_{-2\pi}^{-\pi} e^{2t} \sin(t)\, dt \right. \\
&\qquad + \int_{-3\pi}^{-2\pi} e^{2t} \left(-\sin(t)\right) dt + \int_{-4\pi}^{-3\pi} e^{2t} \sin(t)\, dt \\
&\qquad \left. + \int_{-5\pi}^{-4\pi} e^{2t} \left(-\sin(t)\right) dt + \int_{-6\pi}^{-5\pi} e^{2t} \sin(t)\, dt + \cdots \right] \\
&= 2\sqrt{2}\pi \left[-\int_{-\pi}^{0} e^{2t} \sin(t)\, dt + \int_{-2\pi}^{-\pi} e^{2t} \sin(t)\, dt \right. \\
&\qquad - \int_{-3\pi}^{-2\pi} e^{2t} \sin(t)\, dt + \int_{-4\pi}^{-3\pi} e^{2t} \sin(t)\, dt \\
&\qquad \left. - \int_{-5\pi}^{-4\pi} e^{2t} \sin(t)\, dt + \int_{-6\pi}^{-5\pi} e^{2t} \sin(t)\, dt - \cdots \right]
\end{aligned}$$

For convenience, we'll work out the antiderivative of $e^{2t} \sin(t)$ just once and then plug it into the above. As our first step, we'll use integration by parts, with $u = e^{2t}$ and

$v' = \sin(t)$, so $u' = 2e^{2t}$ and $v = -\cos(t)$.

$$\begin{aligned} \int e^{2t}\sin(t)\,dt &= e^{2t}\left(-\cos(t)\right) - \int 2e^{2t}\left(-\cos(t)\right)\,dt \\ &= -e^{2t}\cos(t) + 2\int e^{2t}\cos(t)\,dt \end{aligned}$$

Here we'll do parts again, this time with $u = e^{2t}$ and $v' = \cos(t)$, so $u' = 2e^{2t}$ and $v = \sin(t)$.

$$\begin{aligned} &= -e^{2t}\cos(t) + 2\left[e^{2t}\sin(t) - \int 2e^{2t}\sin(t)\,dt\right] \\ &= -e^{2t}\cos(t) + 2e^{2t}\sin(t) - 4\int e^{2t}\sin(t)\,dt \end{aligned}$$

Thus

$$\int e^{2t}\sin(t)\,dt = -e^{2t}\cos(t) + 2e^{2t}\sin(t) - 4\int e^{2t}\sin(t)\,dt\,,$$

so

$$5\int e^{2t}\sin(t)\,dt = -e^{2t}\cos(t) + 2e^{2t}\sin(t)\,,$$

from which it follows that

$$\int e^{2t}\sin(t)\,dt = -\frac{1}{5}e^{2t}\cos(t) + \frac{2}{5}e^{2t}\sin(t)\,.$$

We ignore the generic constant because we'll be plugging this antiderivative into definite integrals where the constant would cancel out anyway.

Back to the definite integrals we want to compute:

$$\begin{aligned} &= 2\sqrt{2}\pi\left[-\int_{-\pi}^{0} e^{2t}\sin(t)\,dt + \int_{-2\pi}^{-\pi} e^{2t}\sin(t)\,dt\right. \\ &\qquad - \int_{-3\pi}^{-2\pi} e^{2t}\sin(t)\,dt + \int_{-4\pi}^{-3\pi} e^{2t}\sin(t)\,dt \\ &\qquad \left. - \int_{-5\pi}^{-4\pi} e^{2t}\sin(t)\,dt + \int_{-6\pi}^{-5\pi} e^{2t}\sin(t)\,dt - \cdots\right] \\ &= 2\sqrt{2}\pi\left[-\left(-\frac{1}{5}e^{2t}\cos(t) + \frac{2}{5}e^{2t}\sin(t)\right)\Bigg|_{-\pi}^{0} + \left(-\frac{1}{5}e^{2t}\cos(t) + \frac{2}{5}e^{2t}\sin(t)\right)\Bigg|_{-2\pi}^{-\pi}\right. \\ &\qquad - \left(-\frac{1}{5}e^{2t}\cos(t) + \frac{2}{5}e^{2t}\sin(t)\right)\Bigg|_{-3\pi}^{-2\pi} + \left(-\frac{1}{5}e^{2t}\cos(t) + \frac{2}{5}e^{2t}\sin(t)\right)\Bigg|_{-4\pi}^{-3\pi} \\ &\qquad \left. - \left(-\frac{1}{5}e^{2t}\cos(t) + \frac{2}{5}e^{2t}\sin(t)\right)\Bigg|_{-5\pi}^{-4\pi} + \left(-\frac{1}{5}e^{2t}\cos(t) + \frac{2}{5}e^{2t}\sin(t)\right)\Bigg|_{-6\pi}^{-5\pi} - \cdots\right] \end{aligned}$$

This isn't quite as bad as it looks because $\sin(t) = 0$ whenever any integer multiple of π is plugged in for t. This means that half of the preceding mess can be ignored:

$$\begin{aligned}
&= 2\sqrt{2}\pi \Bigg[-\left(-\frac{1}{5}e^{2t}\cos(t)\right)\Bigg|_{-\pi}^{0} + \left(-\frac{1}{5}e^{2t}\cos(t)\right)\Bigg|_{-2\pi}^{-\pi} \\
&\qquad - \left(-\frac{1}{5}e^{2t}\cos(t)\right)\Bigg|_{-3\pi}^{-2\pi} + \left(-\frac{1}{5}e^{2t}\cos(t)\right)\Bigg|_{-4\pi}^{-3\pi} \\
&\qquad - \left(-\frac{1}{5}e^{2t}\cos(t)\right)\Bigg|_{-5\pi}^{-4\pi} + \left(-\frac{1}{5}e^{2t}\cos(t)+\right)\Bigg|_{-6\pi}^{-5\pi} - \cdots \Bigg] \\
&= \frac{2\sqrt{2}\pi}{5}\Big[e^{2t}\cos(t)\big|_{-\pi}^{0} - e^{2t}\cos(t)\big|_{-2\pi}^{-\pi} \\
&\qquad + e^{2t}\cos(t)\big|_{-3\pi}^{-2\pi} - e^{2t}\cos(t)\big|_{-4\pi}^{-3\pi} \\
&\qquad + e^{2t}\cos(t)\big|_{-5\pi}^{-4\pi} - e^{2t}\cos(t)+\big|_{-6\pi}^{-5\pi} + \cdots \Big] \\
&= \frac{2\sqrt{2}\pi}{5}\Big[\big(e^{0}\cos(0) - e^{-2\pi}\cos(-\pi)\big) - \big(e^{-2\pi}\cos(-\pi) - e^{-4\pi}\cos(-2\pi)\big) \\
&\quad + \big(e^{-4\pi}\cos(-2\pi) - e^{-6\pi}\cos(-3\pi)\big) - \big(e^{-6\pi}\cos(-3\pi) - e^{-8\pi}\cos(-4\pi)\big) \\
&\quad + \big(e^{-8\pi}\cos(-4\pi) - e^{-10\pi}\cos(-5\pi)\big) - \big(e^{-10\pi}\cos(-5\pi) - e^{-12\pi}\cos(-6\pi)\big) \\
&\quad + \cdots \Big]
\end{aligned}$$

Since $e^0 = 1$ and $\cos(0) = 1$, $\cos(-\pi) = -1$, $\cos(-2\pi) = 1$, $\cos(-3\pi) = -1$, and so on, this comes down to:

$$\begin{aligned}
&= \frac{2\sqrt{2}\pi}{5}\Big[\big(1 + e^{-2\pi}\big) - \big(-e^{-2\pi} - e^{-4\pi}\big) \\
&\qquad + \big(e^{-4\pi} + e^{-6\pi}\big) - \big(-e^{-6\pi} - e^{-8\pi}\big) \\
&\qquad + \big(e^{-8\pi} + e^{-10\pi}\big) - \big(-e^{-10\pi} - e^{-12\pi}\big) \\
&\qquad + \cdots \Big] \\
&= \frac{2\sqrt{2}\pi}{5}\Big[\big(1 + e^{-2\pi} + e^{-2\pi} + e^{-4\pi}\big) \\
&\qquad + \big(e^{-4\pi} + e^{-6\pi} + e^{-6\pi} + e^{-8\pi}\big) \\
&\qquad + \big(e^{-8\pi} + e^{-10\pi} + e^{-10\pi} + e^{-12\pi}\big) \\
&\qquad + \cdots \Big] \\
&= \frac{2\sqrt{2}\pi}{5}\Big[1 + 2e^{-2\pi} + 2e^{-4\pi} + 2e^{-6\pi} + 2e^{-8\pi} + 2e^{-10\pi} + 2e^{-12\pi} + \cdots \Big]
\end{aligned}$$

Here we can approximate the answer pretty well by taking the first few terms of the infinite sum – because $e^{-2n\pi} \to 0$ very quickly as $n \to \infty$ – or we can continue by adding up the infinite sum:

$$= \frac{2\sqrt{2}\pi}{5}\left[1 + 2e^{-2\pi} + 2e^{-4\pi} + 2e^{-6\pi} + 2e^{-8\pi} + 2e^{-10\pi} + 2e^{-12\pi} + \cdots\right]$$
$$= \frac{2\sqrt{2}\pi}{5}\left[-1 + 2 + 2e^{-2\pi} + 2\left(e^{-2\pi}\right)^2 + 2\left(e^{-2\pi}\right)^3 + 2\left(e^{-2\pi}\right)^4 + \cdots\right]$$

It is a fact (see Chapter 11 of the text) that the sum of the geometric series $a + ar + ar^2 + ar^3 + \cdots$ is $\frac{a}{1-r}$, so long as $|r| < 1$. In our case, $2\left(e^{-2\pi}\right)^2 + 2\left(e^{-2\pi}\right)^3 + 2\left(e^{-2\pi}\right)^4 + \cdots$ is a geometric series with first term $a = 2$ and common ratio $r = e^{-2\pi} < 1$. It follows that the area of the surface is:

$$= \frac{2\sqrt{2}\pi}{5}\left[-1 + 2 + 2e^{-2\pi} + 2\left(e^{-2\pi}\right)^2 + 2\left(e^{-2\pi}\right)^3 + 2\left(e^{-2\pi}\right)^4 + \cdots\right]$$
$$= \frac{2\sqrt{2}\pi}{5}\left[-1 + \frac{2}{1 - e^{-2\pi}}\right]$$
$$= \frac{2\sqrt{2}\pi}{5}\left[-\frac{1 - e^{-2\pi}}{1 - e^{-2\pi}} + \frac{2}{1 - e^{-2\pi}}\right]$$
$$= \frac{2\sqrt{2}\pi}{5} \cdot \frac{1 + e^{-2\pi}}{1 - e^{-2\pi}}$$

For those who are morbidly curious,

$$\frac{2\sqrt{2}\pi}{5} \cdot [1] \sim 1.777153175$$
$$\frac{2\sqrt{2}\pi}{5} \cdot \left[1 + 2e^{-2\pi}\right] \sim 1.783790638$$
$$\frac{2\sqrt{2}\pi}{5} \cdot \left[1 + 2e^{-2\pi} + 2e^{-4\pi}\right] \sim 1.783803034$$
$$\vdots$$
$$\frac{2\sqrt{2}\pi}{5} \cdot \frac{1 + e^{-2\pi}}{1 - e^{-2\pi}} \sim 1.783803058$$

The first few terms of the series give pretty good approximations of the final answer . . . ■