Mathematics 110 – Calculus of one variable
Trent University 2002-2003

SOLUTIONS TO ASSIGNMENT #10

Series business

Your task, should you choose to undertake it, will be to show that:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

1. Verify the following trigonometric identity. (So long as x is not an integer multiple of π anyway!) *[2]*

$$\frac{1}{\sin^2(x)} = \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{x}{2}\right)} + \frac{1}{\sin^2\left(\frac{x+\pi}{2}\right)}\right]$$

Hint: Use common trig identities and the fact that for any t, $\cos(t) = \sin\left(t + \frac{\pi}{2}\right)$.

Solution. Here goes!

$$\begin{aligned}
\frac{1}{\sin^2(x)} &= \frac{1}{(\sin(x))^2} = \frac{1}{\left(2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right)^2} = \frac{1}{4\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)} \\
&= \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)}\right] = \frac{1}{4}\left[\frac{\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)}{\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)}\right] \\
&= \frac{1}{4}\left[\frac{\sin^2\left(\frac{x}{2}\right)}{\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)} + \frac{\cos^2\left(\frac{x}{2}\right)}{\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)}\right] = \frac{1}{4}\left[\frac{1}{\cos^2\left(\frac{x}{2}\right)} + \frac{1}{\sin^2\left(\frac{x}{2}\right)}\right] \\
&= \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{x}{2}+\frac{\pi}{2}\right)} + \frac{1}{\sin^2\left(\frac{x}{2}\right)}\right] = \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{x}{2}\right)} + \frac{1}{\sin^2\left(\frac{x+\pi}{2}\right)}\right] \qquad \blacksquare
\end{aligned}$$

2. Verify the following trigonometric summation formula for $m \geq 1$. *[2]*

$$1 = \frac{2}{4^m}\sum_{k=0}^{2^{m-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{m+1}}\right)}$$

Hint: Apply the identity from question **1** repeatedly, starting from $1 = \dfrac{1}{\sin^2\left(\frac{\pi}{2}\right)}$. After doing so, you may find the fact that $\sin(t) = \sin(\pi - t)$ comes in handy.

Solution. Note that $\sin\left(\frac{\pi}{2}\right) = 1$, so $1 = \dfrac{1}{1^2} = \dfrac{1}{\sin^2\left(\frac{\pi}{2}\right)}$.

First, using the identity in **1** with $x = \frac{\pi}{2}$, we get:

$$1 = \frac{1}{\sin^2\left(\frac{\pi}{2}\right)} = \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{\pi/2}{2}\right)} + \frac{1}{\sin^2\left(\frac{\pi/2+\pi}{2}\right)}\right] = \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{\pi}{4}\right)} + \frac{1}{\sin^2\left(\frac{3\pi}{4}\right)}\right]$$

Second, using the identity in **1** on each part of the formula above, with $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ respectively, we get (omitting some of the algebra and arithmetic):

$$\begin{aligned}
1 &= \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{\pi}{4}\right)} + \frac{1}{\sin^2\left(\frac{3\pi}{4}\right)}\right] \\
&= \frac{1}{4}\left(\frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{1}{\sin^2\left(\frac{\pi/2+\pi}{8}\right)}\right] + \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{3\pi}{8}\right)} + \frac{1}{\sin^2\left(\frac{\pi/2+3\pi}{8}\right)}\right]\right) \\
&= \frac{1}{4}\left(\frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{1}{\sin^2\left(\frac{5\pi}{8}\right)}\right] + \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{3\pi}{8}\right)} + \frac{1}{\sin^2\left(\frac{7\pi}{8}\right)}\right]\right) \\
&= \frac{1}{16}\left[\frac{1}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{1}{\sin^2\left(\frac{3\pi}{8}\right)} + \frac{1}{\sin^2\left(\frac{5\pi}{8}\right)} + \frac{1}{\sin^2\left(\frac{7\pi}{8}\right)}\right] \\
&= \frac{1}{4^2}\sum_{k=0}^{2^{2+1}-1}\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{2+1}}\right)}
\end{aligned}$$

If we were to continue, at the nth step we would start with the formula from step $n-1$, namely:

$$1 = \frac{1}{4^n}\sum_{k=0}^{2^{n-1}-1}\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^n}\right)}$$

Next, we'd use the identity in **1** on each part of this formula, with $x = \frac{\pi}{2^n}$, $x = \frac{3\pi}{2^n}$, . . . , and $x = \frac{(2^n-1)\pi}{2^n}$ respectively. Applied to $x = \frac{(2k+1)\pi}{2^n}$, this will give:

$$\begin{aligned}
\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^n}\right)} &= \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{(2k+1)\pi/2^n}{2}\right)} + \frac{1}{\sin^2\left(\frac{(2k+1)\pi/2^n+\pi}{2}\right)}\right] \\
&= \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} + \frac{1}{\sin^2\left(\frac{(2k+1)\pi+2^n\pi}{2^{n+1}}\right)}\right] \\
&= \frac{1}{4}\left[\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} + \frac{1}{\sin^2\left(\frac{(2(k+2^{n-1})+1)\pi}{2^{n+1}}\right)}\right]
\end{aligned}$$

It follows that

$$\begin{aligned}
1 &= \frac{1}{4^{n-1}} \sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^n}\right)} \\
&= \frac{1}{4^{n-1}} \sum_{k=0}^{2^{n-1}-1} \frac{1}{4} \left[\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} + \frac{1}{\sin^2\left(\frac{(2(k+2^{n-1})+1)\pi}{2^{n+1}}\right)} \right] \\
&= \frac{1}{4^n} \sum_{k=0}^{2^{n-1}-1} \left[\frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} + \frac{1}{\sin^2\left(\frac{(2(k+2^{n-1})+1)\pi}{2^{n+1}}\right)} \right] \\
&= \frac{1}{4^n} \left[\sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} + \sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2\left(\frac{(2(k+2^{n-1})+1)\pi}{2^{n+1}}\right)} \right] \\
&= \frac{1}{4^n} \left[\sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} + \sum_{k=2^{n-1}}^{2^n-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} \right] \\
&= \frac{1}{4^n} \sum_{k=0}^{2^n-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)}
\end{aligned}$$

A really formal argument would proceed by induction on n.

Unfortunately, what we have so far isn't quite the formula that was asked for. One step more is needed to get there, the key to which is the observation that $\sin(x) = \sin(\pi - x)$*. In particular, for $0 \le k \le 2^n - 1$,

$$\pi - \frac{(2k+1)}{2^{n+1}}\pi = \frac{2^{n+1} - 2k - 1}{2^{n+1}}\pi = \frac{2\left(2^n - k\right) - 1}{2^{n+1}} = \frac{2(\ell+1) - 1}{2^{n+1}},$$

where $0 \le \ell < 2^n - 1$ and $\ell = 2^n - k - 1$. It follows that the terms in the second half of the sum $\frac{1}{4^n} \sum_{k=0}^{2^n-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)}$ repeat the values of the terms in the first half, albeit in reverse order. We can now finish the job:

$$\begin{aligned}
1 &= \frac{1}{4^n} \sum_{k=0}^{2^n-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} = \frac{1}{4^n} \left[\sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} + \sum_{k=2^{n-1}}^{2^n-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} \right] \\
&= \frac{1}{4^n} \left[\sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} + \sum_{\ell=0}^{2^{n-1}-1} \frac{1}{\sin^2\left(\frac{(2\ell+1)\pi}{2^{n+1}}\right)} \right] = \frac{2}{4^n} \sum_{k=0}^{2^{n-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{n+1}}\right)} \quad \blacksquare
\end{aligned}$$

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* You can get this from the addition formula for sin:
$\sin(\pi - x) = \sin(\pi)\cos(-x) + \cos(\pi)\sin(-x) = 0 \cdot \cos(-x) + (-1)\left(-\sin(x)\right) = \sin(x)$

3. Verify the following limit formula, where $k \geq 0$ is fixed. *[2]*

$$\lim_{m\to\infty} 2^m \sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right) = \frac{(2k+1)\pi}{2}$$

Hint: This is really just (a version of) $\lim_{t\to 0} \frac{\sin(t)}{t} = 0 \ldots$

Solution. Recall that $\lim\limits_{t\to 0} \frac{\sin(t)}{t} = 1$.

$$\begin{aligned}
\lim_{m\to\infty} 2^m \sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right) &= \lim_{m\to\infty} \frac{\frac{(2k+1)\pi}{2}}{\frac{(2k+1)\pi}{2}} 2^m \sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right) \\
&= \frac{(2k+1)\pi}{2} \lim_{m\to\infty} \frac{2^m}{\frac{(2k+1)\pi}{2}} \sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right) \\
&= \frac{(2k+1)\pi}{2} \lim_{m\to\infty} \frac{\sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right)}{\frac{(2k+1)\pi}{2^{m+1}}} \\
&= \frac{(2k+1)\pi}{2} \lim_{t\to 0} \frac{\sin(t)}{t} = \frac{(2k+1)\pi}{2} \cdot 1 = \frac{(2k+1)\pi}{2} \qquad \blacksquare
\end{aligned}$$

4. Take the limit as $m \to \infty$ of the identity in **2**, and use **3** to show the following. *[2]*

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

Solution. Here goes!

$$\begin{aligned}
1 = \lim_{m\to\infty} 1 &= \lim_{m\to\infty} \frac{2}{4^m} \sum_{k=0}^{2^{m-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{m+1}}\right)} = \lim_{m\to\infty} \sum_{k=0}^{2^{m-1}-1} \frac{2}{4^m \sin^2\left(\frac{(2k+1)\pi}{2^{m+1}}\right)} \\
&= \lim_{m\to\infty} \sum_{k=0}^{2^{m-1}-1} \frac{2}{2^{2m} \sin^2\left(\frac{(2k+1)\pi}{2^{m+1}}\right)} = \lim_{m\to\infty} \sum_{k=0}^{2^{m-1}-1} \frac{2}{\left[2^m \sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right)\right]^2} \\
&= \sum_{k=0}^{\infty} \lim_{m\to\infty} \frac{2}{\left[2^m \sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right)\right]^2} = \sum_{k=0}^{\infty} \frac{2}{\lim_{m\to\infty} \left[2^m \sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right)\right]^2} \\
&= \sum_{k=0}^{\infty} \frac{2}{\left[\lim_{m\to\infty} 2^m \sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right)\right]^2} = \sum_{k=0}^{\infty} \frac{2}{\left[\frac{(2k+1)\pi}{2}\right]^2} \\
&= \sum_{k=0}^{\infty} \frac{2}{\frac{(2k+1)^2\pi^2}{2^2}} = \sum_{k=0}^{\infty} \frac{8}{(2k+1)^2\pi^2} = \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}
\end{aligned}$$

Multiplying through by $\frac{\pi^2}{8}$, it follows that $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$. ■

5. Use **4** and some algebra to check that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

is true. *[2]*

Hint: Split up $\sum_{n=1}^{\infty} \frac{1}{n^2}$ into the sums of the terms for even and odd n respectively and try to rewrite the sum of the terms for even n.

Solution. First, following up on the hint:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \left[\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \cdots\right] + \left[\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \cdots\right] \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} + \sum_{k=0}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{8} + \sum_{k=0}^{\infty} \frac{1}{4k^2} = \frac{\pi^2}{8} + \frac{1}{4} \sum_{k=0}^{\infty} \frac{1}{k^2}\end{aligned}$$

It follows that $\left(1 - \frac{1}{4}\right) \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$, so $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \cdot \frac{\pi^2}{8} = \frac{\pi^2}{6}$.

Bonus. A major assumption has been made without proper justification in one of the steps outlined above. What is it? *[1]*

Solution. The tricky part is in **4**. Taking the limit of the partial sums at the same time as one takes the limit of the individual terms, *i.e.* the step

$$\lim_{m \to \infty} \sum_{k=0}^{2^{m-1}-1} \frac{2}{\left[2^m \sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right)\right]^2} = \sum_{k=0}^{\infty} \lim_{m \to \infty} \frac{2}{\left[2^m \sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right)\right]^2}$$

in the solution above, needs some justification. This procedure can be justified, but it takes either an appeal to a fairly sophisticated result (*e.g.* Tannery's Theorem) or some *ad hoc* argument that it safe to do this in this case. ■

Reference *(This is where we stole the argument!)*

1. Josef Hofbauer, *A Simple Proof of* $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$ *and Related Identities*, *American Mathematical Monthly*, Volume 109, Number 2, February 2002, pp. 196–200.