Mathematics 110 – Calculus of one variable

Trent University 2002-2003

ASSIGNMENT #10

Due: Monday, 7 April, 2003

Series business

Your task, should you choose to undertake it, will be to show that:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

1. Verify the following trigonometric identity. (So long as x is not an integer multiple of π anyway!) *[2]*

$$\frac{1}{\sin^2(x)} = \frac{1}{4}\left(\frac{1}{\sin^2\left(\frac{x}{2}\right)} + \frac{1}{\sin^2\left(\frac{\pi+x}{2}\right)}\right)$$

Hint: Use common trig identities and the fact that for any t, $\cos(t) = \sin\left(t + \frac{\pi}{2}\right)$.

2. Verify the following trigonometric summation formula for $m \geq 1$. *[2]*

$$1 = \frac{2}{4^m} \sum_{k=0}^{2^{m-1}-1} \frac{1}{\sin^2\left(\frac{(2k+1)\pi}{2^{m+1}}\right)}$$

Hint: Apply the identity from question **1** repeatedly, starting from $1 = \dfrac{1}{\sin^2\left(\frac{\pi}{2}\right)}$.

3. Verify the following limit formula, where $k \geq 0$ is fixed. *[2]*

$$\lim_{m\to\infty} 2^m \sin\left(\frac{(2k+1)\pi}{2^{m+1}}\right) = \frac{(2k+1)\pi}{2}$$

Hint: This is really just (a version of) $\lim_{t\to 0} \frac{\sin(t)}{t} = 0 \ldots$

4. Take the limit as $m \to \infty$ of the identity in **2**, and use **3** to show the following. *[2]*

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

5. Use **4** and some algebra to check that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

is true. *[2]*

Hint: Split up $\sum_{n=1}^{\infty} \frac{1}{n^2}$ into the sums of the terms for even and odd n respectively and try to rewrite the sum of the terms for even n.

Bonus. A major assumption has been made without proper justification in one of the steps outlined above. What is it? *[1]*