Mathematics 110 – Calculus of one variable
Trent University 2001-2002

SOLUTIONS TO ASSIGNMENT #8

Length of curves

1. Find the length of the curve given by the paramateric equations

$$x = t\cos(t)$$
$$y = t\sin(t)$$

for $0 \leq t \leq 2\pi$. *[4]*

Solution. The length of the curve is given by the formula, which can be found in §10.3, $\int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$. Since $\frac{dx}{dt} = \cos(t) - t\sin(t)$ and $\frac{dy}{dt} = \sin(t) + t\cos(t)$, the length of the given curve can be computed as follows:

$$\int_0^{2\pi} \sqrt{(\cos(t) - t\sin(t))^2 + (\sin(t) + t\cos(t))}\, dt$$
$$= \int_0^{2\pi} \sqrt{\cos^2(t) - 2t\sin(t)\cos(t) + t^2\sin^2(t) + \sin^(t) + 2t\cos(t)\sin(t) + t^2\cos^2(t)}\, dt$$
$$= \int_0^{2\pi} \sqrt{\cos^2(t) + t^2\sin^2(t) + \sin^(t) + t^2\cos^2(t)}\, dt$$
$$= \int_0^{2\pi} \sqrt{(1+t^2)\cos^2(t) + (1+t^2)\sin^2(t)}\, dt$$
$$= \int_0^{2\pi} \sqrt{(1+t^2)\left(\sin^2(t) + \cos^2(t)\right)}\, dt$$
$$= \int_0^{2\pi} \sqrt{1+t^2}\, dt \qquad \text{since } \sin^2(t) + \cos^2(t) = 1$$

Using $t = \tan(\theta)$, $dt = \sec^2(\theta)\, d\theta$, now gives:

$$= \int_{t=0}^{t=2\pi} \sqrt{1 + \tan^2(\theta)}\ \sec^2(\theta) d\theta$$
$$= \int_{t=0}^{t=2\pi} \sqrt{\sec^2(\theta)}\ \sec^2(\theta)\, d\theta$$
$$= \int_{t=0}^{t=2\pi} \sec^3(\theta) d\theta$$

This is an integral we've seen before, so we'll cut to the chase:

$$
\begin{aligned}
=&\frac{1}{2}\tan(\theta)\sec(\theta)\big|_{t=0}^{t=2\pi} + \frac{1}{2}\ln\left(\tan(\theta)+\sec(\theta)\right)\big|_{t=0}^{t=2\pi} \\
=&\frac{1}{2}\cdot t\cdot\sqrt{1+t^2}\Big|_0^{2\pi} + \frac{1}{2}\ln\left(t+\sqrt{1+t^2}\right)\Big|_0^{2\pi} \\
=&\frac{1}{2}\cdot 2\pi\cdot\sqrt{1+(2\pi)^2} - \frac{1}{2}\cdot 0\cdot\sqrt{1+0^2} \\
&\quad + \frac{1}{2}\ln\left(2\pi+\sqrt{1+(2\pi)^2}\right) - \frac{1}{2}\ln\left(0+\sqrt{1+0^2}\right) \\
=&\pi\sqrt{1+4\pi^2} - 0 + \frac{1}{2}\ln\left(2\pi+\sqrt{1+4\pi^2}\right) - 0 \\
=&\pi\sqrt{1+4\pi^2} + \frac{1}{2}\ln\left(2\pi+\sqrt{1+4\pi^2}\right)
\end{aligned}
$$

Ugly answer! ■

2. Find the length of the curve in three dimensions given by the paramateric equations

$$
\begin{aligned}
x &= 3\cos(t) \\
y &= 3\sin(t) \\
z &= 4t
\end{aligned}
$$

for $0 \le t \le 2\pi$. *[4]*

Solution. The length of a parametric curve in three dimensions is given by a formula very similar to that used in **2**, namely $\int_0^{2\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$. Things do work out rather more nicely here, though:

$$
\begin{aligned}
&\int_0^{2\pi}\sqrt{(-3\sin(t))^2+(3\cos(t))^2+4^2}\,dt \\
=&\int_0^{2\pi}\sqrt{9\sin^2(t)+9\cos^2(t)+16}\,dt \\
=&\int_0^{2\pi}\sqrt{9\left(\sin^2(t)+\cos^2(t)\right)+16}\,dt \\
=&\int_0^{2\pi}\sqrt{9+16}\,dt \\
=&\int_0^{2\pi}\sqrt{25}\,dt \\
=&\int_0^{2\pi}5\,dt \\
=&\ 5t\big|_0^{2\pi} \\
=&5\cdot 2\pi - 5\cdot 0 \\
=&10\pi \qquad ■
\end{aligned}
$$

3. Find, if you can, the length of the perimeter of the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$. *[2]*

Solution. One way to try this is to parametrize the ellipse and try to use the same formula used in **2**. One nice parametrization of the given ellipse is

$$x = 2\cos(t)$$
$$y = 3\sin(t)$$

where $0 \leq t \leq 2\pi$. Unfortunately, the resulting integral,

$$\begin{aligned}
&\int_0^{2\pi} \sqrt{(-2\sin(t))^2 + (3\cos(t))^2}\, dt \\
&= \int_0^{2\pi} \sqrt{4\sin^2(t) + 9\cos^2(t)}\, dt \\
&= \int_0^{2\pi} \sqrt{4\left(\sin^2(t) + \cos^2(t)\right) + 5\cos^2(t)} \\
&= \int_0^{2\pi} \sqrt{4 + 5\cos^2(t)}\, dt
\end{aligned}$$

is rather intractable ...

The options on how to deal with this include:

i. Approximate the integral numerically using Riemann sums or some variation of them.

ii. Find a function which is easier to integrate that is close to $\sqrt{4 + 5\cos^2(t)}$ and use its integral to approximate the given one.

iii. Express $\sqrt{4 + 5\cos^2(t)}$ as a power series, integrate that term by term, and (probably) get an answer in terms of an infinite series.

iv. Look it up ...

No matter how you sliced it, good luck! ■