Mathematics 1121H – Calculus II
Trent University, Winter 2026

Solutions to Assignment #4
Series Business
Due on Friday, 6 February.

Consider the *alternating harmonic series*, $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$. This series *converges*, *i.e.* adds up to a real number, which really means that $\displaystyle\lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\right]$ exists and is the real number the series adds up to.

1. Use a suitable `for` loop in SageMath to discover the value of n required to ensure that the first two decimal places of $\displaystyle\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$ are the same as for the sum of the whole alternating harmonic series. *[2]*

Solution. The brutally simple idea is to repeatedly add the next term of the series, print out the partial sum in decimal, and experiment a bit to see when the first two places don't change anymore.

```
[1]: # 1
var('k')
var('n')
var('s')
n = 200
s = 0
for k in [1..n]:
    s = s + (-1)^(k+1)/k
    print( k, N(s) )
```

```
1 1.00000000000000
2 0.500000000000000
3 0.833333333333333
4 0.583333333333333
5 0.783333333333333
6 0.616666666666667
7 0.759523809523809
8 0.634523809523809
9 0.745634920634921
10 0.645634920634921
```

Skipping ahead . . .

```
156 0.689952324990583
157 0.696321751742175
158 0.689992637818125
159 0.696281945994225
160 0.690031945994225
161 0.696243126118449
162 0.690070286612276
163 0.696205255937429
164 0.690107694961820
165 0.696168301022426
```

So it seems that we need $n = 159$ to ensure that the first two decimal places settle down in the partial sums. □

2. What is the sum of the alternating harmonic series? Why? *[2]*

SOLUTION. The sum of the alternating harmonic series is $\ln(2)$, which has a decimal approximation of:

```
[2]: # 2
     N(log(2))
```

```
[2]: 0.693147180559945
```

Why is the sum $\ln(2)$? Here is a very verbose answer.

Consider the following geometric series with $a = 1$ and $r = x$:

$$1 - x + x^2 - x^3 + x^4 - \cdots = \frac{1}{1-(-x)} = \frac{1}{1+x}$$

Note that this series converges when $|r| = |x| < 1$, *i.e.* when $-1 < x < 1$. Integrating the geomteric series above term-by-term gives

$$\int \left(1 - x + x^2 - x^3 + x^4 - \cdots\right) dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \cdots,$$

and integrating its sum gives

$$\int \frac{1}{1+x}\,dx = \int \frac{1}{w}\,dw \quad \text{[Substituting } w = 1+x\text{, so } dw = dx.\text{]}$$
$$= \ln(w) + C = \ln(1+x) + C.$$

We thus have

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = textln(1+x) + C.$$

Plugging in $x = 0$ gives us

$$0 - \frac{0^2}{2} + \frac{0^3}{3} - \frac{0^4}{4} + \cdots = 0 = textln(1+0) + C = 0 + C,$$

so $C = 0$ and thus

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = textln(1+x).$$

It turns out that this new series converges at one more point than its parent series, namely for $-1 < x \le 1$. Plugging in $x = 1$ gives us

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = 1 - \frac{1^2}{2} + \frac{1^3}{3} - \frac{1^4}{4} + \cdots$$
$$= \ln(1+1) = \ln(2), \text{ as claimed. } \square$$

Consider *Gregory's series*, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$. This series also converges. (To a different sum than the alternating harmonic series does, mind you.)

3. Use a suitable `for` loop in SageMath to discover the value of n required to ensure that the first two decimal places of $\sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1}$ are the same as for the sum of all of Gregory's series. *[4]*

SOLUTION. Same idea as in the solution to question **1** ...

```
[3]: # 3
     clear_vars
     var('k')
     var('n')
     var('s')
     n = 100
     s = 0
     for k in [1..n]:
         s = s + (-1)^(k+1)/(2*k-1)
         print( k, N(s) )
```

```
1 1.00000000000000
2 0.666666666666667
3 0.866666666666667
```

Skipping ahead ...

```
52 0.780590915382685
53 0.790114724906494
54 0.780768930513971
55 0.789943242440577
56 0.780934233431568
57 0.789783790953691
58 0.781088138779778
59 0.789635147326787
60 0.781231785982249
```

So it seems that we need $n = 54$ to ensure that the first two decimal places settle down in the partial sums. □

4. What is the sum of Gregory's series? Why? *[2]*

SOLUTION. The sum of Gregory's series is $\frac{\pi}{4}$, which has a decimal approximation of:

```
[4]: # 4
     N(pi/4)
```

```
[4]: 0.785398163397448
```

We can see why the sum of Gregory's series is $\frac{\pi}{4}$ through reasoning similar to that used in the solution to question **2**, presented here is a slightly less verbose fashion:

$$\begin{aligned}
\frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots \\
\implies \int \frac{1}{1+x^2}\,dx &= \int \left(1 - x^2 + x^4 - x^6 + x^8 - \cdots\right) dx \\
\implies \arctan(x) &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + C \\
\text{When } x = 0, \text{ we get } \arctan(0) &= 0 = 0 - \frac{0^3}{3} + \frac{0^5}{5} - \frac{0^7}{7} + \frac{0^9}{9} - \cdots + C, \\
\text{so } C = 0, \text{ and thus } \arctan(x) &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots .
\end{aligned}$$

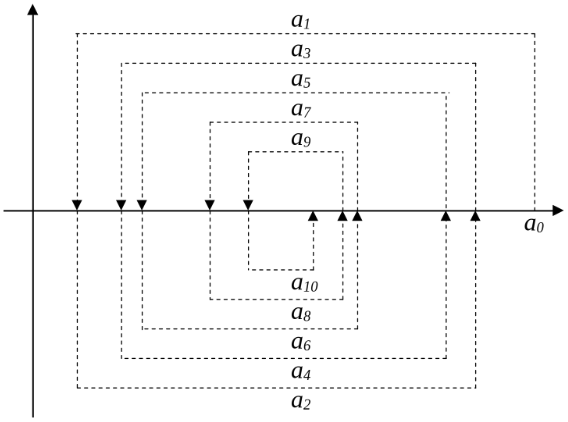
Similarly to the situation in question **2**, while the original geometric series converges only when $-1 < x < 1$, its integral converges when $-1 < x \leq 1$. Plugging in $x = 1$, we get

$$\frac{\pi}{4} = \arctan(1) = 1 - \frac{1^3}{3} + \frac{1^5}{5} - \frac{1^7}{7} + \cdots = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots, \text{ as claimed. } \square$$

Both the alternating harmonic series and Gregory's series are example of *alternating series*: series that add up non-zero terms a_k that are alternately positive and negative, *i.e.* such that $a_k < 0$ if and only if $a_{k+1} > 0$.

5. Suppose $\sum_{k=1}^{\infty} a_k$ is an alternating series such that $|a_{k+1}| < |a_k|$ for all k and that $\lim_{k\to\infty} |a_k| = 0$. Explain why the series has to converge. [You need not give an actual proof.] *[2]*

SOLUTION. Explanation by picture! Keep in mind the requirements that the series is alternating, that $|a_{k+1}| < |a_k|$ for all k and that $\lim_{k\to\infty} |a_k| = 0$. The picture is for the case that $a_0 > 0$; the case where $a_0 < 0$ works similarly on the other side of the y-axis.



$\square$