Mathematics 1121H – Calculus II

TRENT UNIVERSITY, Winter 2026

Assignment #3

Due on Friday, 30 January.[*]

The *Cantor set* C is defined by the following process:

Step 0. Start with the closed unit interval $[0, 1]$.

Step 1. Remove the open middle third of the interval, leaving $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$.

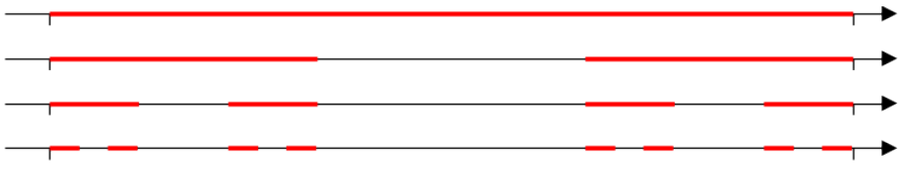
Step 2. Remove the open middle third of each of the remaining intervals, leaving $\left[0, \frac{1}{9}\right]$, $\left[\frac{2}{9}, \frac{3}{9}\right]$, $\left[\frac{6}{9}, \frac{7}{9}\right]$ and $\left[\frac{8}{9}, 1\right]$.

$\vdots$

Step n. Remove the open middle third of each of the intervals remaining after step $n-1$, leaving $\left[0, \frac{1}{3^n}\right]$, $\left[\frac{2}{3^n}, \frac{3}{3^n}\right]$, $\ldots$, $\left[\frac{3^n-1}{3^n}, 1\right]$

$\vdots$

Here is a picture of Steps 0 through 3:



C, the Cantor set, is the limit of the process, *i.e.* what remains after infinitely many steps. Note that C is not empty; for example, $\frac{1}{3} \in C$.

1. What is the total length of the intervals removed from $[0, 1]$ to make C? *[2]*

HINT. The sum of a *geometric series* $\displaystyle\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$ is $\dfrac{a}{1-r}$, as long as the *common ratio* r satisfies $|r| < 1$.

2. Suppose that $f(x) = \begin{cases} 1 & x \in C \\ 0 & x \notin C \end{cases}$. What should $\displaystyle\int_0^1 f(x)\,dx$ be? Why? [You need not give an actual proof.] *[2]*

3. Suppose that $g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. Prove that $g(x)$ is not Riemann integrable on $[0, 1]$. *[4]*

4. Despite the fact that $g(x)$ is not Riemann integrable on $[0, 1]$, what should $\displaystyle\int_0^1 g(x)\,dx$ be? Why? *[2]*

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[*] You should submit your solutions via Blackboard's Assignments module, preferably as a single pdf. If submission via Blackboard fails, please submit your work to your instructor by email or on paper as soon as you can. You may work together, look things up, and use whatever tools you like, so long as you *write up your submission by yourself* and give due credit to your collaborators and any sources and tools you actually used.