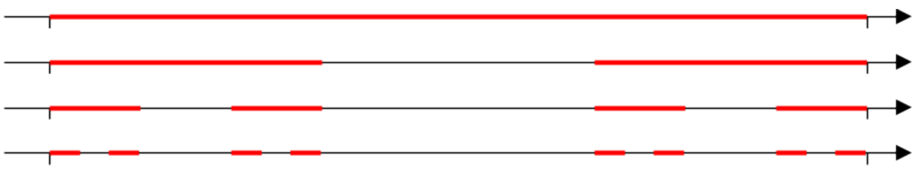
Mathematics 1121H – Calculus II
Trent University, Winter 2026

Solutions to Assignment #3
Due on Friday, 30 January.

The *Cantor set* C is defined by the following process:

Step 0. Start with the closed unit interval $[0,1]$.
Step 1. Remove the open middle third of the interval, leaving $\left[0,\frac{1}{3}\right]$ and $\left[\frac{2}{3},1\right]$.
Step 2. Remove the open middle third of each of the remaining intervals, leaving $\left[0,\frac{1}{9}\right]$, $\left[\frac{2}{9},\frac{3}{9}\right]$, $\left[\frac{6}{9},\frac{7}{9}\right]$ and $\left[\frac{8}{9},1\right]$.
⋮
Step n. Remove the open middle third of each of the intervals remaining after step $n-1$, leaving $\left[0,\frac{1}{3^n}\right]$, $\left[\frac{2}{3^n},\frac{3}{3^n}\right]$, ... , $\left[\frac{3^n-1}{3^n},1\right]$
⋮

Here is a picture of Steps 0 through 3:



C, the Cantor set, is the limit of the process, *i.e.* what remains after infinitely many steps. Note that C is not empty; for example, $\frac{1}{3} \in C$.

1. What is the total length of the intervals removed from $[0,1]$ to make C? *[2]*

Hint. The sum of a *geometric series* $\displaystyle\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$ is $\dfrac{a}{1-r}$, as long as the *common ratio* r satisfies $|r| < 1$.

Solution. At step 1, we remove an interval of length $\frac{1}{3}$ from the interval $[0,1]$; at step 2, we remove two intervals of length $\frac{1}{9}$; at step 3, we remove four intervals of length $\frac{1}{27}$; and so on for a total length of $\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\frac{8}{81}+\cdots$. It's not hard to see that this is a geometric series with first term $a=\frac{1}{3}$ and common ration $r=\frac{2}{3}$, and $\left|\frac{2}{3}\right|<1$, so the total length of the intervals removed in the process of creating the Cantor set is

$$\text{length}\left([0,1]\setminus C\right) = \frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\frac{8}{81}+\cdots = \frac{\frac{1}{3}}{1-\frac{2}{3}} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1. \quad \square$$

2. Suppose that $f(x) = \begin{cases} 1 & x \in C \\ 0 & x \notin C \end{cases}$. What should $\displaystyle\int_0^1 f(x)\,dx$ be? Why? [You need not give an actual proof.] *[2]*

Solution. We should have $\int_0^1 f(x)\,dx = 0$. The interval $[0,1]$ has length 1 and the total length of the intervals removed from $[0,1]$ by the solution to question **1** is 1, so the total "length" of the Cantor set C ought to be $1-1=0$. We then ought to have something like

$$\int_0^1 f(x)\,dx = \int_C 1\,dx + \int_{[0,1]\setminus C} 0\,dx = 1\cdot\text{length}(C) + 0\cdot\text{length}\left([0,1]\setminus C\right) = 1\cdot 0 + 0\cdot 1 = 0. \quad \square$$

3. Suppose that $g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. Prove that $g(x)$ is not Riemann integrable on $[0,1]$. *[4]*

Solution. Recall that between any two real numbers we can find a rational number and also an irrational number. It follows that between any two real numbers $g(x)$ takes on the value 1 and also – at some other point – takes on the value 0. This, in turn, means that for any partition $P = \{t_0, t_1, t_2, \dots t_n\}$ of $[0,1]$, we have

$$\begin{aligned} m_i &= \inf\{g(x) \mid t_{i-1} \le x \le t_i\} = 0 \\ \text{and } M_i &= \sup\{g(x) \mid t_{i-1} \le x \le t_i\} = 1. \end{aligned}$$

Thus, for every partition P of $[0,1]$,

$$\begin{aligned} L(f,P) &= m_1(t_1-t_0) + m_2(t_2-t_1) + \cdots + m_n(t_n-t_{n-1}) \\ &= 0(t_1-t_0) + 0(t_2-t_1) + \cdots + 0(t_n-t_{n-1}) = 0 \\ \text{and } U(f,P) &= M_1(t_1-t_0) + M_2(t_2-t_1) + \cdots + M_n(t_n-t_{n-1}) \\ &= 1(t_1-t_0) + 1(t_2-t_1) + \cdots + 1(t_n-t_{n-1}) = t_n - t_0 = 1-0 = 1. \end{aligned}$$

This means that for all $\varepsilon > 0$ with $\varepsilon < 1$ and all partitions P of $[0,1]$, we have

$$U(f,P) - L(f,P) = 1 - 0 = 1 \not< \varepsilon.$$

By Theorem 4 of the handout *Darboux's Version of the Riemann Integral* it follows that $g(x)$ is not integrable on $[0,1]$. $\square$

4. Despite the fact that $g(x)$ is not Riemann integrable on $[0,1]$, what should $\int_0^1 g(x)\,dx$ be? Why? *[2]*

Solution. We should have $\int_0^1 g(x)\,dx = 0$. The reason is similar to that in the solution to question **2**: the "length" of all of $\mathbb{Q}$, never mind $\mathbb{Q} \cap [0,1]$, is 0. This is a consequence of $\mathbb{Q}$ being *countable*, meaning that it is possible to list all the rational numbers in a list indexed by the natural numbers $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. (Ask about how to check this if you haven't seen it before!) Suppose $q_0, q_1, q_2, q_3, \dots$ is such a list of all the rational numbers.

Given any $\varepsilon > 0$, we can enclose all the rational numbers in open intervals whose combined length is $\leq \varepsilon$ as follows:

Enclose q_0 in $\left(q_0 - \frac{\varepsilon}{4}, q_0 + \frac{\varepsilon}{4}\right)$, which has length $\frac{\varepsilon}{2}$.
Enclose q_1 in $\left(q_1 - \frac{\varepsilon}{8}, q_1 + \frac{\varepsilon}{8}\right)$, which has length $\frac{\varepsilon}{4}$.
Enclose q_2 in $\left(q_2 - \frac{\varepsilon}{16}, q_2 + \frac{\varepsilon}{16}\right)$, which has length $\frac{\varepsilon}{8}$.

$\vdots$

Enclose q_n in $\left(q_n - \frac{\varepsilon}{2^{n+2}}, q_n + \frac{\varepsilon}{2^{n+2}}\right)$, which has length $\frac{\varepsilon}{2^{n+1}}$.

$\vdots$

The total length of all of these open intervals is

$$\frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \cdots + \frac{\varepsilon}{2^{n+1}} + \cdots,$$

which is a geometric series with first term $\frac{\varepsilon}{2}$ and common ratio $r = \frac{1}{2}$. Since $\left|\frac{1}{2}\right| < 1$, it follows that the total length is $\dfrac{\frac{\varepsilon}{2}}{1 - \frac{1}{2}} = \dfrac{\frac{\varepsilon}{2}}{\frac{1}{2}} = \varepsilon$. Since many of these intervals must actually overlap [Why?], their combined length is actually $< \varepsilon$.

Since the rational numbers can be enclosed in a collection of intervals with a combined length $< \varepsilon$ for every ε, the "length" of all of $\mathbb{Q}$, and hence also of $[0,1] \cap \mathbb{Q}$, must be 0. By reasoning similar to that in the solution to question **2**, it follows that

$$\begin{aligned}\int_0^1 g(x)\,dx &= \int_{[0,1]\cap\mathbb{Q}} 1\,dx + \int_{[0,1]\setminus\mathbb{Q}} 0\,dx \\ &= 1 \cdot \text{length}([0,1] \cap \mathbb{Q}) + 0 \cdot \text{length}\left([0,1] \setminus \mathbb{Q}\right) \\ &= 1 \cdot 0 + 0 \cdot 1 = 0. \quad \square\end{aligned}$$