Mathematics 1121H – Calculus II
Trent University, Winter 2026

Solution to Assignment #1
Due on Friday, 16 January.

Please read the handout *Darboux's Version of the Riemann Integral* before tackling this assignment. (Much of the material will be developed in lecture as well, so you might want to come to class. :-)

1. Prove Theorem 4 of the handout. *[10]*

Solution. Here is the theorem we are asked to prove:

> **Theorem 4.** Suppose $f(x)$ is bounded on $[a,b]$. Then $f(x)$ is integrable on $[a,b]$ if and only if for every $\varepsilon > 0$ there is a partition P of $[a,b]$ such that
> $$U(f,P) - L(f,P) < \varepsilon.$$

We are interested mainly in the "if" direction, since that is the one we are more likely to use. However, we will prove both directions of the "if and only if" for completeness (and possible bonus marks :-).

Proof. Suppose $f(x)$ is bounded on $[a,b]$.

($\Leftarrow$) Assume that for each $\varepsilon > 0$ there is a partition P of $[a,b]$ such that $U(f,P)-L(f,P) < \varepsilon$. Observe that for any such partition P of $[a,b]$,

$$\begin{gathered}\inf\{\,U(f,Q) \mid Q \text{ is a partition of } [a,b]\,\} \leq U(f,P)\\ \text{and } L(f,P) \leq \sup\{\,L(f,Q) \mid Q \text{ is a partition of } [a,b]\,\},\end{gathered}$$

the former because $U(f,P)$ is one of the upper sums for which the infimum is a lower bound, and the latter because $L(f,P)$ is one of the lower sums for which the supremum is an upper bound. It follows that

$$\begin{aligned}&\inf\{\,U(f,Q) \mid Q \text{ is a partition of } [a,b]\,\} - \sup\{\,L(f,Q) \mid Q \text{ is a partition of } [a,b]\,\}\\ &\leq U(f,P) - L(f,P) < \varepsilon.\end{aligned}$$

On the other hand, because every upper sum is larger than every lower sum, we also have

$$\inf\{\,U(f,Q) \mid Q \text{ is a partition of } [a,b]\,\} \geq \sup\{\,L(f,Q) \mid Q \text{ is a partition of } [a,b]\,\}$$

by Corollary 3 of the handout. Combining these, we see that for any $\varepsilon > 0$,

$$0 \leq \inf\{\,U(f,Q) \mid Q \text{ is a partition of } [a,b]\,\} - \sup\{\,L(f,Q) \mid Q \text{ is a partition of } [a,b]\,\} < \varepsilon,$$

which is only possible if

$$\inf\{\,U(f,Q) \mid Q \text{ is a partition of } [a,b]\,\} - \sup\{\,L(f,Q) \mid Q \text{ is a partition of } [a,b]\,\} = 0,$$

so $\inf\{\,U(f,Q) \mid Q \text{ is a partition of } [a,b]\,\} = \sup\{\,L(f,Q) \mid Q \text{ is a partition of } [a,b]\,\}$. By the definition of integrability, this means that $f(x)$ is integrable on $[a,b]$. $\square$

For completeness (and bonus points :-), we also prove the other direction, though it less likely to be used.

($\Rightarrow$) Assume that $f(x)$ is integrable on $[a,b]$. By definition, this means that

$$\inf \{ U(f,Q) \mid Q \text{ is a partition of } [a,b] \} = \sup \{ L(f,Q) \mid Q \text{ is a partition of } [a,b] \}.$$

Suppose now that we are handed some $\varepsilon > 0$. $\inf \{ U(f,Q) \mid Q \text{ is a partition of } [a,b] \}$ is the *greatest* lower bound for upper sums of $f(x)$ on partitions of $[a,b]$, so there must be some partition R of $[a,b]$ such that $U(f,R) - \inf \{ U(f,Q) \mid Q \text{ is a partition of } [a,b] \} < \frac{\varepsilon}{2}$. Similarly, $\sup \{ L(f,Q) \mid Q \text{ is a partition of } [a,b] \}$ is the *least* upper bound for lower sums of $f(x)$ on partitions of $[a,b]$, so there must be some partition S of $[a,b]$ such that $\sup \{ L(f,Q) \mid Q \text{ is a partition of } [a,b] \} - L(f,S) < \frac{\varepsilon}{2}$. Let $P = R \cup S$. Then $U(f,P) \leq U(f,R)$ and $L(f,P) \geq L(f,S)$ by Lemma 1 of the handout, and it follows that

$$\begin{aligned} U(f,P) - L(f,P) &\leq U(f,R) - L(f,S) = U(f,R) - 0 + L(f,S) \\ &= U(f,R) - \inf \{ U(f,Q) \mid Q \text{ is a partition of } [a,b] \} \\ &\quad + \sup \{ L(f,Q) \mid Q \text{ is a partition of } [a,b] \} - L(f,S) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

exploiting the fact that

$$- \inf \{ U(f,Q) \mid Q \text{ is a partition of } [a,b] \} + \sup \{ L(f,Q) \mid Q \text{ is a partition of } [a,b] \} = 0$$

because the infimum and the supremum are equal.

Thus if $f(x)$ is integrable on $[a,b]$, then for every $\varepsilon > 0$ there is a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$. ■