An Integral Tale

or

How do we compute $\displaystyle\int_0^\pi \frac{x\sin(x)}{1+\cos^2(x)}\,dx$ *?*

I ran across this integral* while trying to find an example to present in class the other year that did not need advanced integration techniques, but was still quite difficult. (I did not end up presenting it in class ... :-)

False start the first.

The derivative of $\cos(x)$ is $-\sin(x)$, and there is a $\sin(x)$ available as a factor of the numerator, so perhaps we can use the substitution $w = \cos(x)$ to simplify the integrand. Then $dw = -\sin(x)\,dx$, so $(-1)\,dw = \sin(x)\,dx$ and, if we change the limits as we go along, we have $\begin{matrix} x & 0 & \pi \\ w & 1 & -1 \end{matrix}$. The one hitch is that x is the other factor of the numerator: there is nothing we can do with it except solve for x in $w = \cos(x)$, so $x = \arccos(w)$. Ugh! Let's give it a shot anyway:

$$\int_0^\pi \frac{x\sin(x)}{1+\cos^2(x)}\,dx = \int_1^{-1} \frac{\arccos(w)}{1+w^2}\,(-1)\,dw = \int_{-1}^1 \frac{\arccos(w)}{1+w^2}\,dw$$

The last step uses the general property of definite integrals that switching the the limits switches the sign: $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

At this point the options seem to be to simplify the $\arccos(w)$ by doing a substitution, which would probably get us back to where we started, or to try integration by parts, which seems more promising because $\arccos(w)$ and $\dfrac{1}{1+w^2}$ are pretty dissimilar. Since most of us don't know the antiderivative of $\arccos(w)$ off the top of our heads, but maybe remember the derivative, we'll use the parts $u = \arccos(w)$ and $v' = \dfrac{1}{1+w^2}$. Then $u' = \dfrac{d}{dw}\arccos(w) = \dfrac{-1}{\sqrt{1-w^2}}$ and $v = \arctan(w)$, which gives us:

$$\int_{-1}^1 \frac{\arccos(w)}{1+w^2}\,dw = \arccos(w)\arctan(w)\big|_{-1}^1 - \int_{-1}^1 \frac{-\arctan(w)}{\sqrt{1-w^2}}\,dw$$

The square root in the denominator of the integral remaining on the right makes it likely that that integral is at least as hard as the one we started with on the left, so this entire approach is more than likely going nowhere.

* It is Exercise 93 in §5.5 of *Calculus*, 8th Edition (Early Transcendentals), by James Stewart.

False start the second.

Since x is so dissimilar from the other factor, $\dfrac{\sin(x)}{1+\cos^2(x)}$, of the integrand, perhaps it would be better to go for integration by parts right away. The question then is how to divide up these dissimilar factors between u and v' in the integration by parts formula $\displaystyle\int_a^b u \cdot v' \, dx = u \cdot v|_a^b - \int_a^b u' \cdot v \, dx$.

If we try $u = x$ and $v' = \dfrac{\sin(x)}{1+\cos^2(x)}$, we get $u' = \frac{d}{dx}x = 1$, which is a promising simplification, but to get v we have to put in some work. We'll use the substitution $w = \cos(x)$, so $dw = -\sin(x)\,dx$ and thus $\sin(x)\,dx = (-1)\,dw$, to compute $v = \int v'\,dx$:

$$v = \int \frac{\sin(x)}{1+\cos^2(x)}\,dx = \int \frac{1}{1+w^2}\,(-1)\,dw = -\arctan(w) = -\arctan\left(\cos(x)\right)$$

Plugging all this into the integration by parts formula gives us:

$$\int_0^\pi \frac{x\sin(x)}{1+\cos^2(x)}\,dx = x\left[-\arctan\left(\cos(x)\right)\right]\Big|_0^\pi - \int_0^\pi 1 \cdot \left[-\arctan\left(\cos(x)\right)\right]\,dx$$

$\arctan\left(\cos(x)\right)$ doesn't look like it's easier to handle than the original problem (feel free to give it a try!), so perhaps it would be better to break up the product in the opposite way.

If we try $u = \dfrac{\sin(x)}{1+\cos^2(x)}$ and $v' = x$, then it is easy to see that $v = \dfrac{x^2}{2}$. We have to work a little harder to get u', though, using the Chain, Quotient, and Power Rules:

$$\begin{aligned} u' &= \frac{d}{dx}\left(\frac{\sin(x)}{1+\cos^2(x)}\right) = \frac{\left[\frac{d}{dx}\sin(x)\right]\cos^2(x) - \sin(x)\left[\frac{d}{dx}\cos^2(x)\right]}{\left[\cos^2(x)\right]^2} \\ &= \frac{\cos(x)\cdot\cos^2(x) - \sin(x)\cdot 2\cos(x)\left(-\sin(x)\right)}{\cos^4(x)} \\ &= \frac{\cos^2(x) + 2\sin^2(x)}{\cos^3(x)} = \frac{1+\sin^2(x)}{\cos^3(x)} = \sec^3(x) + \tan^2(x)\sec(x) \end{aligned}$$

Trig identities being what they are, there are lots of ways to write u'. However we do so, though, what we get after plug things into the integration by parts formula isn't very promising either. For example, if we use the second form on the last line, we get:

$$\int_0^\pi \frac{x\sin(x)}{1+\cos^2(x)}\,dx = \frac{\sin(x)}{1+\cos^2(x)}\cdot\frac{x^2}{2}\bigg|_0^\pi - \int_0^\pi \frac{1+\sin^2(x)}{\cos^3(x)}\cdot\frac{x^2}{2}\,dx$$

Time to try something different ...

A method that works.

Following a hint given in a preceding problem for a related integral, I finally tried to use the substitution $u = \pi - x$. Then $x = \pi - u$, $du = (-1)\,dx$, $dx = (-1)\,du$, and, changing the limits as we go along, $\begin{matrix} x & 0 & \pi \\ u & \pi & 0 \end{matrix}$.

The key to making this substitution work is the fact that $\sin(\pi - x) = \sin(x)$ and $\cos(\pi - x) = -\cos(x)$. Why do these identities work? They do due to the periodic nature of $\sin(x)$ and $\cos(x)$: shifting the graph of either to the left or right by π radians will give you the negative of the original graph, *i.e.* $\sin(x \pm \pi) = -\sin(x)$ and $\cos(x \pm \pi) = -\cos(x)$. Since $\sin(x)$ is an odd function, *i.e.* $\sin(-x) = -\sin(x)$ for all x, it follows that $\sin(\pi - x) = \sin(-x + \pi) == -\sin(-x) = -(-\sin(x)) = \sin(x)$. Similarly, since $\cos(x)$ is an even function, *i.e.* $\cos(-x) = \cos(x)$ for all x, it follows that $\cos(\pi - x) = \cos(-x + \pi) = -\cos(-x) = -\cos(x)$.

Let's go for it, using the given substitution and the identities above:

$$\int_0^\pi \frac{x\sin(x)}{1+\cos^2(x)}\,dx = \int_\pi^0 \frac{(\pi - u)\sin(\pi - u)}{1+\cos^2(\pi - u)}\,(-1)\,du = \int_0^\pi \frac{(\pi - u)\sin(\pi - u)}{1+\cos^2(\pi - u)}\,du$$

(The last by the general fact that $\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$.)

$$= \int_0^\pi \frac{(\pi - u)\sin(u)}{1+[-\cos(u)]^2}\,du = \int_0^\pi \frac{(\pi - u)\sin(u)}{1+\cos^2(u)}\,du$$

$$= \int_0^\pi \left[\frac{\pi\sin(u)}{1+\cos^2(u)} - \frac{u\sin(u)}{1+\cos^2(u)}\right]du$$

$$= \int_0^\pi \frac{\pi\sin(u)}{1+\cos^2(u)}\,du - \int_0^\pi \frac{u\sin(u)}{1+\cos^2(u)}\,du$$

$$= \int_0^\pi \frac{\pi\sin(x)}{1+\cos^2(x)}\,dx - \int_0^\pi \frac{x\sin(x)}{1+\cos^2(x)}\,dx$$

The last step is by the general fact that $\int_a^b f(x)\,dx = \int_a^b f(u)\,du$. (Think about it: $\int_a^b f(t)\,dt$ is a number; t, or whatever you choose to call the variable inside the integral, is there just for bookkeeping.) Observe now that the integral $\displaystyle\int_0^\pi \frac{x\sin(x)}{1+\cos^2(x)}\,dx$ occurs at both ends of the calculation. That is, we have the equation:

$$\int_0^\pi \frac{x\sin(x)}{1+\cos^2(x)}\,dx = \int_0^\pi \frac{\pi\sin(x)}{1+\cos^2(x)}\,dx - \int_0^\pi \frac{x\sin(x)}{1+\cos^2(x)}\,dx$$

Moving $\displaystyle\int_0^\pi \frac{x\sin(x)}{1+\cos^2(x)}\,dx$ to the left-hand side gives

$$2\int_0^\pi \frac{x\sin(x)}{1+\cos^2(x)}\,dx = \int_0^\pi \frac{\pi\sin(x)}{1+\cos^2(x)}\,dx\,,$$

and dividing by 2 on both sides gives

$$\int_0^\pi \frac{x\sin(x)}{1+\cos^2(x)}\,dx = \frac{1}{2}\int_0^\pi \frac{\pi\sin(x)}{1+\cos^2(x)}\,dx = \frac{\pi}{2}\int_0^\pi \frac{\sin(x)}{1+\cos^2(x)}\,dx\,.$$

It remains to finish the job by evaluating the integral on the right. We will use the substitution $w = \cos(x)$. (Yes, the one from the first false start and the first try in the second false start. Sigh.) Then we have $dw = -\sin(x)\,dx$, so $(-1)\,dw = \sin(x)\,dx$ and, changing the limits as we go along, we also have $\begin{matrix} x & 0 & \pi \\ w & 1 & -1 \end{matrix}$.

$$\begin{aligned}
\int_0^\pi \frac{x\sin(x)}{1+\cos^2(x)}\,dx &= \frac{\pi}{2}\int_0^\pi \frac{\sin(x)}{1+\cos^2(x)}\,dx = \frac{\pi}{2}\int_1^{-1} \frac{1}{1+w^2}\,(-1)\,dw \\
&= \frac{\pi}{2}\int_{-1}^{1} \frac{1}{1+w^2}\,dw = \frac{\pi}{2}\arctan(w)\Big|_{-1}^{1} \\
&= \frac{\pi}{2}\arctan(1) - \frac{\pi}{2}\arctan(-1) = \frac{\pi}{2}\cdot\frac{\pi}{4} - \frac{\pi}{2}\cdot\left(-\frac{\pi}{4}\right) \\
&= \frac{\pi^2}{8} + \frac{\pi^2}{8} = \frac{\pi^2}{4}
\end{aligned}$$

Whew! ■