Trent University, Winter 2024

MATH 1120H

~~Midterm Test~~

Solutions to Assignment $\#\pi + e$

Tuesday, 27 February
11:00-11:50

Name: Hero Zero

Student Number: 0000000

Question	Mark
1	____
2	____
3	____
Total	____ /30

Instructions

- *Show all your work.* Legibly, please! Simplify where you reasonably can.
- *If you have a question, ask it!*
- Use the back sides of all the pages for rough work or extra space.
- You may use a calculator and (all sides of) an A4- or letter-size aid sheet.

Note. *Technically, this is an extra assignment which, should you choose to do it, will go into the pool from which the best ten are chosen to count towards the final mark.*

1. Compute any *four* (4) of integrals **a**–**f**. *[12 = 4 × 3 each]*

a. $\displaystyle\int_0^{\pi/4} \sin(x)\sec^3(x)\,dx$ **b.** $\displaystyle\int_0^1 \ln(y)\,dy$ **c.** $\displaystyle\int \frac{1}{z^3+z}\,dz$

d. $\displaystyle\int (r^2+1)^{-1/2}\,dr$ **e.** $\displaystyle\int \frac{e^s}{e^{2s}+1}\,ds$ **f.** $\displaystyle\int_1^{\infty} \frac{1}{t^3}\,dt$

SOLUTIONS. **a.** We will use the substitution $u = \cos(x)$, so $du = (-1)\sin(x)\,dx$, and change the limits as we go along, $\begin{array}{ccc} x & 0 & \pi/4 \\ u & 1 & 1/\sqrt{2} \end{array}$. Here we go:

$$\begin{aligned}
\int_0^{\pi/4} \sin(x)\sec^3(x)\,dx &= \int_0^{\pi/4} \frac{\sin(x)}{\cos^3(x)}\,dx = \int_1^{1/\sqrt{2}} \frac{1}{u^3}(-1)\,du = \int_{1/\sqrt{2}}^1 \frac{1}{u^3}\,du \\
&= \int_{1/\sqrt{2}}^1 u^{-3}\,du = \left.\frac{u^{-2}}{-2}\right|_{1/\sqrt{2}}^1 = \left.\frac{-1}{2u^2}\right|_{1/\sqrt{2}}^1 \\
&= \left(\frac{-1}{2\cdot 1^2}\right) - \left(\frac{-1}{2\cdot\left(\frac{1}{\sqrt{2}}\right)^2}\right) = \left(-\frac{1}{2}\right) - \left(-\frac{1}{1}\right) = \frac{1}{2} \quad \square
\end{aligned}$$

b. Note that $\ln(y) \to -\infty$ as $y \to 0^+$, so this is an improper integral which will have to be evaluated using a limit. Along the way we will use integration by parts, with $u = \ln(y)$ and $v' = 1$, so $u' = \dfrac{1}{y}$ and $v = y$, and l'Hôpital's Rule. Here we go:

$$\begin{aligned}
\int_0^1 \ln(y)\,dy &= \lim_{t\to 0^+} \int_t^1 \ln(y)\,dy = \lim_{t\to 0^+} \left[\ln(y)\cdot y\big|_t^1 - \int_t^1 \frac{1}{y}\cdot y\,dy \right] \\
&= \lim_{t\to 0^+} \left[1\cdot\ln(1) - t\cdot\ln(t) - \int_t^1 1\,dy \right] \\
&= \lim_{t\to 0^+} \left[1\cdot 0 - \frac{\ln(t)}{\frac{1}{t}} - y\big|_t^1 \right] = \lim_{t\to 0^+} \left[0 - \frac{\ln(t)}{\frac{1}{t}} - (1-t) \right] \\
&= \left(-\lim_{t\to 0^+} \frac{\ln(t) \to -\infty}{\frac{1}{t} \quad \to +\infty} \right) - \left(\lim_{t\to 0^+} [1-t] \right) \\
&= \left(-\lim_{t\to 0^+} \frac{\frac{d}{dt}\ln(t)}{\frac{d}{dt}\left(\frac{1}{t}\right)} \right) - (1-0) = \left(-\lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{-1}{t^2}} \right) - 1 \\
&= \left(-\lim_{t\to 0^+} \frac{1}{t}\cdot\frac{t^2}{-1} \right) - 1 = \left(-\lim_{t\to 0^+} (-t) \right) - 1 = -(-0) - 1 = -1 \quad \square
\end{aligned}$$

c. This is a job for partial fraction decomposition. Since 1 has lower degree than z^3+z, we can skip ahead to factoring the polynomial, which is fortunately very easy here: $z^3+z = z\left(z^2+1\right)$. Note that $z^2+1 \geq 1 > 0$ for all z, so we can't factor it in turn.

This factorization means that the integrand can be rewritten as a sum of partial fractions as follows:

$$\frac{1}{z^3+z} = \frac{1}{z\left(z^2+1\right)} = \frac{A}{z} + \frac{Bz+C}{z^2+1} = \frac{A\left(z^2+1\right)+(Bz+C)z}{z\left(z^2+1\right)} = \frac{(A+B)z^2+Cz+A}{z\left(z^2+1\right)}$$

Since the coefficients of the same powers of z in the numerator at each end must be equal, it follows that $A+B=0$, $C=0$, and $A=1$, from which it follows in turn that $B=-1$. The integrand therefore decomposes as:

$$\frac{1}{z^3+z} = \frac{1}{z} + \frac{-1}{z^2+1} = \frac{1}{z} - \frac{1}{z^2+1}$$

We can now – finally! – actually integrate:

$$\begin{aligned}\int \frac{1}{z^3+z}\,dz &= \int \left(\frac{1}{z} - \frac{1}{z^2+1}\right) dz = \int \frac{1}{z}\,dz - \int \frac{1}{z^2+1}\,dz \\ &= \ln(z) - \arctan(z) + C \quad \square\end{aligned}$$

d. We will use the trigonometric substitution $r=\tan(\theta)$, so $dr=\sec^2(\theta)\,d\theta$.

$$\begin{aligned}\int (r^2+1)^{-1/2}\,dr &= \int \left(\tan^2(\theta)+1\right)^{-1/2}\sec^2(\theta)\,d\theta = \int \left(\sec^2(\theta)\right)^{-1/2}\sec^2(\theta)\,d\theta \\ &= \int \frac{1}{\sqrt{\sec^2(\theta)}}\sec^2(\theta)\,d\theta = \int \frac{\sec^2(\theta)}{\sec(\theta)}\,d\theta \\ &= \int \sec(\theta)\,d\theta = \ln\left(\sec(\theta)+\tan(\theta)\right)+C = \ln\left(\sqrt{r^2+1}+r\right)+C,\end{aligned}$$

since $\tan(\theta)=r$ and $\sec(\theta)=\sqrt{\sec^2(\theta)}=\sqrt{\tan^2(\theta)+1}=\sqrt{r^2+1}$. $\square$

e. We will use the substitution $w=e^s$, so $dw=e^s\,ds$. Here goes:

$$\int \frac{e^s}{e^{2s}+1}\,ds = \int \frac{e^s}{\left(e^s\right)^2+1}\,ds = \int \frac{1}{w^2+1}\,dw = \arctan(w)+C = \arctan\left(e^s\right)+C \quad \square$$

f. Note that since we have a limit of integration of ∞, this is an improper integral which ought to be evaluated using a limit.

$$\begin{aligned}\int_1^\infty \frac{1}{t^3}\,dt &= \lim_{k\to\infty}\int_1^k t^{-3}\,dt = \lim_{k\to\infty}\left[\left.\frac{t^{-2}}{-2}\right|_1^k\right] = \lim_{k\to\infty}\left[\left.\frac{-1}{2t^2}\right|_1^k\right] = \lim_{k\to\infty}\left[\frac{-1}{2k^2} - \frac{-1}{2\cdot 1^2}\right] \\ &= \lim_{k\to\infty}\left[\frac{-1}{2k^2}+\frac{1}{2}\right] = -0+\frac{1}{2} = \frac{1}{2} \quad \blacksquare\end{aligned}$$

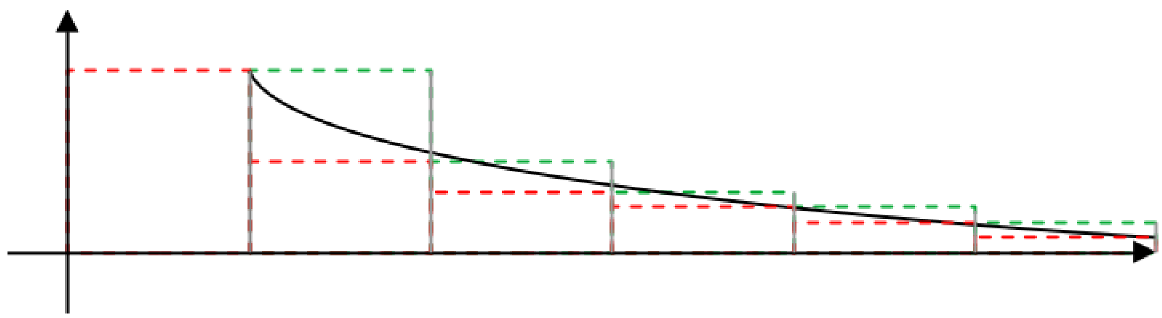
2. Do any *two* (2) of parts **a**–**c**. *[8 = 2 × 4 each]*

a. Explain why the infinite sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ ought to add up to a finite value.

b. Find the area between the curves $y = \sqrt{x}$ and $y = x^3$ for $0 \le x \le 1$.

c. Find the centroid of the of the diamond-shaped region whose corners are $(0, 1)$, $(-1, 0)$, $(0, -1)$, and $(1, 0)$. (You may assume a constant density of 1.)

SOLUTIONS. **a.** This is like question **3** on Assignment #4. Consider the curve $y = \dfrac{1}{x^3}$ from $x = 1$ onwards, as in the diagram below.



Considering the areas of the rectangles whose tops are the dashed lines under the graph of the function, for each $n \ge 1$ we have $\dfrac{1}{(n+1)^3} < \displaystyle\int_n^{n+1} \frac{1}{x^3}\, dx$. It follows that

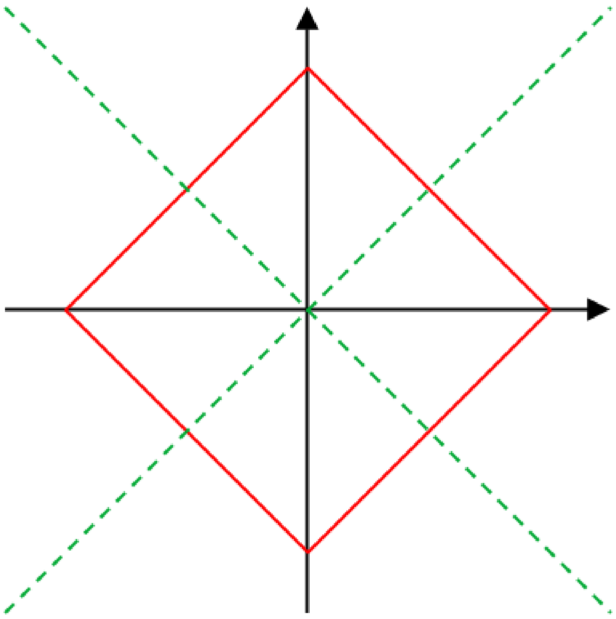
$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^3} &= \frac{1}{1^3} + \sum_{n=2}^{\infty} \frac{1}{n^3} = 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^3} \\
&< 1 + \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^3}\, dx = 1 + \int_1^{\infty} \frac{1}{x^3}\, dx = 1 + \int_1^{\infty} x^{-3}\, dx \\
&= 1 + \lim_{k\to\infty} \int_1^k x^{-3}\, dx = 1 + \lim_{k\to\infty} \left.\frac{x^{-2}}{-2}\right|_1^k = 1 + \lim_{k\to\infty} \left.\frac{-1}{2x^2}\right|_1^k \\
&= 1 + \lim_{k\to\infty} \left[\frac{-1}{2k^2} - \frac{-1}{2\cdot 1^2}\right] = 1 + \lim_{k\to\infty} \left[\frac{-1}{2k^2} + \frac{1}{2}\right] = 1 + 0 + \frac{1}{2} = \frac{3}{2}
\end{aligned}$$

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ should add up to some finite value less than $\dfrac{3}{2} = 1.5$. □

b. Note that the two curves intersect at $x = 0$ and at $x = 1$. but at no other points: $\sqrt{x} = x^3 \Rightarrow x = x^6$ and $x \ge 0 \Rightarrow x = 0$ or $x = 1$. Between $x = 0$ and $x = 1$ the curve $y = \sqrt{x}$ is above the curve $y = x^3$, *e.g.* $\sqrt{1/2} \approx 0.7 > 0.125 = (1/2)^3$. It follows that the area between the curves is given by:

$$\begin{aligned}
\text{Area} &= \int_0^1 (\text{upper} - \text{lower})\, dx = \int_0^1 \left(\sqrt{x} - x^3\right)\, dx = \int_0^1 \left(x^{1/2} - x^3\right)\, dx \\
&= \left(\frac{x^{3/2}}{3/2} - \frac{x^4}{4}\right)\Bigg|_0^1 = \left(\frac{2x^{3/2}}{3} - \frac{x^4}{4}\right)\Bigg|_0^1 = \left(\frac{2 \cdot 1^{3/2}}{3} - \frac{1^4}{4}\right) - \left(\frac{2 \cdot 0^{3/2}}{3} - \frac{0^4}{4}\right) \\
&= \frac{2}{3} - \frac{1}{4} - 0 = \frac{8}{12} - \frac{3}{12} = \frac{5}{12} \approx 0.4167 \quad \square
\end{aligned}$$

c. Here's a picture of the region:



Without calculus. Note that both of the axes, as well as the lines $y = -x$ and $y = x$ are lines about which the region is symmetric. It follows that the centroid must be on each of these lines. Since the only point that is on all of these lines is the origin, the centroid must be the origin, *i.e.* $(\bar{x}, \bar{y}) = (0, 0)$. $\square$

"Calculus! Lots and lots of calculus!"† We compute M, M_y, and M_x on the way to finding the coordinates of the centroid of the given region.

M. With a constant density of 1, the mass M of the region is equal to its area. Observe that for $-1 \leq x \leq 0$, the upper border of the region is given by $y = x + 1$ and the lower border is given by $y = -x - 1$, while for $0 \leq x \leq 1$, the upper border of the region is given by $y = -x + 1$ and the lower border is given by $y = x - 1$. It follows that:

† N_o^e in *The Mathtrix*.

$$
\begin{aligned}
M = \text{Area} &= \int_{-1}^{1} (\text{upper} - \text{lower})\; dx \\
&= \int_{-1}^{0} ((x+1) - (-x-1))\; dx + \int_{0}^{1} ((-x+1) - (x-1))\; dx \\
&= \int_{-1}^{0} (2x+2)\, dx + \int_{0}^{1} (-2x+1)\, dx = \left(x^2 + 2x\right)\Big|_{-1}^{0} + \left(-x^2 + 2x\right)\Big|_{0}^{1} \\
&= \left(0^2 + 2 \cdot 0\right) - \left((-1)^2 + 2(-1)\right) + \left(-1^2 + 2 \cdot 1\right) - \left(-0^2 + 2 \cdot 0\right) \\
&= 0 - (-1) + 1 - 0 = 1 + 1 = 2
\end{aligned}
$$

M_y. With a constant density of 1, the moment M_y of the region about the y-axis is given by integrating x times the length of the vertical cross-section (*i.e.* upper $-$ lower) over the region. Note that for $-1 \le x \le 0$, the length of the vertical cross-section at x is $(x+1) - (-x-1) = 2x+2$, and for $0 \le x \le 1$, the length of the vertical cross-section at x is $(-x+1) - (x-1) = -2x+2$. It follows that:

$$
\begin{aligned}
M_y &= \int_{-1}^{1} x\,(\text{upper} - \text{lower})\; dx = \int_{-1}^{0} x(2x+2)\, dx + \int_{0}^{1} x(-2x+2)\, dx \\
&= \int_{-1}^{0} \left(2x^2 + 2x\right)\; dx + \int_{0}^{1} \left(-2x^2 + 2x\right)\; dx = \left(\frac{2x^3}{3} + x^2\right)\Bigg|_{-1}^{0} + \left(-\frac{2x^3}{3} + x^2\right)\Bigg|_{0}^{1} \\
&= \left(\frac{2 \cdot 0^3}{3} + 0^2\right) - \left(\frac{2(-1)^3}{3} + (-1)^2\right) + \left(-\frac{2 \cdot 1^3}{3} + 1^2\right) - \left(-\frac{2 \cdot 0^3}{3} + 0^2\right) \\
&= 0 - \left(\frac{-2}{3} + 1\right) + \left(-\frac{2}{3} + 1\right) - 0 = -\frac{1}{3} + \frac{1}{3} = 0
\end{aligned}
$$

M_x. With a constant density of 1, the moment M_x of the region about the x-axis is given by integrating y times the length of the horizontal cross-section (*i.e.* right $-$left) over the region. Note that for $-1 \le y \le 0$, the length of the horizontal cross-section at y is $(y+1) - (-y-1) = 2y+2$, and for $0 \le y \le 1$, the length of the horizontal cross-section at y is $(-y+1) - (y-1) = -2y+2$. It follows that:

$$
\begin{aligned}
M_x &= \int_{-1}^{1} x\,(\text{right} - \text{left})\; dx = \int_{-1}^{0} y(2y+2)\, dy + \int_{0}^{1} y(-2y+2)\, dy \\
&= \int_{-1}^{0} \left(2y^2 + 2y\right)\; dy + \int_{0}^{1} \left(-2y^2 + 2y\right)\; dy = \left(\frac{2y^3}{3} + y^2\right)\Bigg|_{-1}^{0} + \left(-\frac{2y^3}{3} + y^2\right)\Bigg|_{0}^{1} \\
&= \left(\frac{2 \cdot 0^3}{3} + 0^2\right) - \left(\frac{2(-1)^3}{3} + (-1)^2\right) + \left(-\frac{2 \cdot 1^3}{3} + 1^2\right) - \left(-\frac{2 \cdot 0^3}{3} + 0^2\right) \\
&= 0 - \left(\frac{-2}{3} + 1\right) + \left(-\frac{2}{3} + 1\right) - 0 = -\frac{1}{3} + \frac{1}{3} = 0
\end{aligned}
$$

The centroid at last. It follows from all of the above that the coordinates of the centroid of the given region are:

$$
(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{0}{2}, \frac{0}{2}\right) = (0, 0) \quad \blacksquare
$$

3. Do *one* (1) of parts **a** or **b**. *[10]*

a. Find the arc-length of the curve $y = \ln(\cos(x))$, where $0 \leq x \leq \frac{\pi}{4}$.

b. Compute $\int x^2 \arctan(x)\, dx$.

SOLUTIONS. **a.** We compute the derivative ...

$$\frac{dy}{dx} = \frac{d}{dx}\ln(\cos(x)) = \frac{1}{\cos(x)} \cdot \frac{d}{dx}\cos(x) = \frac{1}{\cos(x)} \cdot (-\sin(x)) = -\tan(x)$$

... and then plug it into the arc-length formula:

$$\begin{aligned} \text{arc-length} &= \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\pi/4} \sqrt{1 + (-\tan x)^2}\, dx = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx \\ &= \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/4} \sec(x)\, dx = \ln(\sec(x) + \tan(x))\Big|_0^{\pi/4} \\ &= \ln(\sec(\pi/4) + \tan(\pi/4)) - \ln(\sec(0) + \tan(0)) \\ &= \ln\left(\sqrt{2} + 1\right) - \ln(1 + 0) = \ln\left(\sqrt{2} + 1\right) - 0 = \ln\left(\sqrt{2} + 1\right) \approx 0.8814 \quad \square \end{aligned}$$

b. We will start by using integration by parts with $u = \arctan(x)$ and $v' = x^2$, so $u' = \frac{1}{1+x^2}$ and $v = \frac{x^3}{3}$. This will be followed up with a touch of algebraic trickery and substitution.

$$\begin{aligned} \int x^2 \arctan(x)\, dx &= \frac{1}{3}x^3 \arctan(x) - \int \frac{1}{3} \cdot \frac{x^3}{1+x^2}\, dx = \frac{1}{3}x^3 \arctan(x) - \frac{1}{3}\int \frac{x \cdot x^2}{1+x^2}\, dx \\ &= \frac{1}{3}x^3 \arctan(x) - \frac{1}{3}\int \frac{x \cdot \left(1 + x^2 - 1\right)}{1+x^2}\, dx \\ &= \frac{1}{3}x^3 \arctan(x) - \frac{1}{3}\left[\int \frac{x \cdot \left(1+x^2\right)}{1+x^2}\, dx - \int \frac{x}{1+x^2}\, dx\right] \\ &= \frac{1}{3}x^3 \arctan(x) - \frac{1}{3}\int x\, dx + \frac{1}{3}\int \frac{x}{1+x^2}\, dx \end{aligned}$$

Substitute $w = 1 + x^2$, so $dw = 2w\, dw$ and $x\, dx = \frac{1}{2}\, dw$.

$$\begin{aligned} &= \frac{1}{3}x^3 \arctan(x) - \frac{1}{3} \cdot \frac{x^2}{2} + \frac{1}{3}\int w\, \frac{1}{2}\, dw \\ &= \frac{1}{3}x^3 \arctan(x) - \frac{x^2}{6} + \frac{1}{6} \cdot \frac{w^2}{2} + C \\ &= \frac{1}{3}x^3 \arctan(x) - \frac{x^2}{6} + \frac{\left(1+x^2\right)^2}{12} + C \quad \blacksquare \end{aligned}$$

NOTE. The alternative to the algebraic trickery used above to decompose $\frac{x^3}{1+x^2}$ is to do long division of $1 + x^2$ into x^3, which is a bit more tedious.

[Total = 30]