Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2024

Solutions to Assignment #8
Calculating π

1. Verify that the series $\displaystyle\sum_{n=0}^{\infty} \frac{2}{(4n+1)(4n+3)}$ converges using one or more of the convergence tests given in class. *[2]*

SOLUTION. There are several ways to do this. One of the simplest is to use the Basic Comparison Test. From $n = 1$ on, we have

$$0 \le \frac{2}{(4n+1)(4n+3)} = \frac{2}{16n^2+16n+3} = \frac{1}{8n^2+8n+\frac{3}{2}} < \frac{1}{n^2},$$

since $8n^2 + 8n + \frac{3}{2} > n^2$. As $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-Test because it has $p = 2 > 1$ (or by question **3** on Assignment #4), it follows by the Basic Comparison Test that $\displaystyle\sum_{n=0}^{\infty} \frac{2}{(4n+1)(4n+3)}$ converges as well. $\square$

NOTE. One could also use the Generalized p-Test, for something even simpler, or the Integral Test, for something a little harder, among the tests that we have seen in class.

2. Use SageMath to to find the sum of the series in **1**. *[1]*

SOLUTION. Here we go:

```
[1]: var('n')
     sum( 2/((4*n+1)*(4*n+3)), n, 0, oo )

[1]: 1/4*pi
```

That is, $\displaystyle\sum_{n=0}^{\infty} \frac{2}{(4n+1)(4n+3)} = \frac{\pi}{4}$. $\square$

3. What series involving powers of x has $\dfrac{1}{1+x^2}$ as its sum? For which values of x does this series converge? *[3]*

SOLUTION. Observe that $\dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)}$. The latter version has the form of the sum of a geometric series with $a = 1$ and $r = -x^2$, so

$$\begin{aligned}\frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} = \frac{a}{1-r} \\ &= \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots \\ &= 1 + 1\left(-x^2\right) + \left(-x^2\right)^2 + 1\left(-x^2\right)^3 + \cdots \\ &= 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}.\end{aligned}$$

A geometric series (with $a \neq 0$) converges exactly when when the common ratio r has $|r| < 1$. In this case, it means that the series we obtained above converges exactly when $|r| = \left|-x^2\right| = x^2 < 1$, *i.e.* exactly when $-1 < x < 1$. $\square$

NOTE. Observe that while the expression $\frac{1}{1+x^2}$ is defined for all $x \in \mathbb{R}$, the series it is the sum of, $\sum_{n=0}^{\infty} (-1)^n x^{2n}$, converges only for $-1 < x < 1$. This kind of mismatch is a frequent problem when working with *power series*, that is, series involving powers of x.

4. Since $\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$ what series involving powers of x should be equal to $\arctan(x)$ when it converges? For which values of x does this series converge? *[3]*

Hint: This series converges for almost, but not quite, the same values of x that the series in **3** does.

SOLUTION. Well, integration is the reverse operation to integration, so ...

$$\begin{aligned}
\arctan(x) &= \int \frac{1}{1+x^2}\, dx = \int \left(\sum_{n=0}^{\infty} (-1)^n x^{2n} \right) dx = \int \left(1 - x^2 + x^4 - x^6 + \cdots\right) dx \\
&= \int 1\, dx - \int x^2\, dx + \int x^4\, dx - \int x^6\, dx + \cdots = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \\
&= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}
\end{aligned}$$

Since $\arctan(x) = 0$ and $x^{2n+1} = 0$ for all $n \geq 0$ when $x = 0$, it follows that $C = 0$, and so:

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

It remains to determine for which values of x this series converges. Observe that when $|x| > 1$, we have

$$\lim_{n\to\infty} \left| \frac{(-1)^n x^{2n+1}}{2n+1} \right| = \lim_{n\to\infty} \frac{|x|^{2n+1}}{2n+1} = \infty \neq 0$$

because exponential growth beats polynomial growth. Since $\lim_{n\to\infty} \left| \frac{(-1)^n x^{2n+1}}{2n+1} \right| \neq 0$, we must have $\lim_{n\to\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \neq 0$. This means, by the Divergence Test, that $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ diverges when $|x| > 1$, *i.e.* when $x < -1$ or when $x > 1$.

On the other hand, suppose that $|x| < 1$. In this case,

$$\lim_{n\to\infty} \left| \frac{(-1)^n x^{2n+1}}{2n+1} \right| = \lim_{n\to\infty} \frac{|x|^{2n+1}}{2n+1} \begin{matrix} \to 0 \\ \to \infty \end{matrix} = 0,$$

so the Divergence Test is silent on whether the series converges or not. However, since we also have that the series alternates between positive and negative values because of the $(-1)^n$ component of the numerator, and $\left| \frac{(-1)^n x^{2n+1}}{2n+1} \right| = \frac{|x|^{2n+1}}{2n+1}$ is non-increasing when $|x| < 1$, the Alternating Series Test tells us that $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ converges when $|x| < 1$.

It remains to check what happens when $x = \pm 1$. We could apply the Alternating Series Test to these borderline cases too, but, being lazy, we hand the problem off to SageMath:

```
[2]: sum( (-1)^n/(2*n+1), n, 0, oo )
```

```
[2]: 1/4*pi
```

```
[3]: sum( (-1)^n*(-1)^(2*n+1)/(2*n+1), n, 0, oo )
```

```
[3]: -1/4*pi
```

Thus $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ converges for $x = \pm 1$.

Putting all of this together, we see that $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ converges exactly when $-1 \leq x \leq 1$, and diverges when $x < -1$ or $x > 1$. □

5. Given that $\arctan(1) = \frac{\pi}{4}$, what is the connection betwen the series in **1** and **4**?

SOLUTION. Well, the series for $\arctan(x)$ with $x = 1$ and the series in **1** both add up to $\frac{\pi}{4}$:

$$\begin{aligned}\frac{\pi}{4} = \arctan(1) &= \sum_{n=0}^{\infty} \frac{(-1)^n 1^{2n+1}}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{2}{(4n+1)(4n+3)} = \frac{2}{3} + \frac{2}{35} + \frac{2}{99} + \cdots\end{aligned}$$

There is a deeper connection, though. Since $\displaystyle\frac{2}{(4n+1)(4n+3)} = \frac{1}{4n+1} - \frac{1}{4n+3}$,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{2}{(4n+1)(4n+3)} &= \sum_{n=0}^{\infty} \left[\frac{1}{4n+1} - \frac{1}{4n+3}\right] \\ &= \left[\frac{1}{1} - \frac{1}{3}\right] + \left[\frac{1}{5} - \frac{1}{7}\right] + \left[\frac{1}{9} - \frac{1}{11}\right] + \cdots \\ &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.\end{aligned}$$

That is, consecutive terms of the series for $\arctan(1)$ are a partial fraction decomposition of the terms of the series in **1**, so the two series are basically different forms of the same thing. ■

NOTE. The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ is called *Gregory's Series* in most modern textbooks. The earliest known version of this series is credited to the Indian mathematician and astronomer Mādhava of Sangamagrāma (*c.* 1340-1425), but it was rediscovered several times, including by the Scottish mathematician and astronomer James Gregory (1638-1675) in 1671 and the German polymath Gottfried Wilhelm Leibniz (1646-1716) in 1673. Leibniz, along with Isaac Newton (1642-1727), is credited with inventing modern calculus.