Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2022

Solutions to Assignment #9
Power Series
Due on Friday, 25 March.

Please show all your work. As with all the assignments in this course, unless stated otherwise on the assignment, you are permitted to work together and look things up, so long as you acknowledge the sources you used and the people you worked with.

Please read §11.4 in the textbook and review the lecture on the Alternating Series Test and conditional *vs.* absolute convergence before tackling tis assignment.

1. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ln(n)}{n} = \frac{\ln(1)}{1} - \frac{\ln(2)}{2} + \frac{\ln(3)}{3} - \frac{\ln(4)}{4} + \cdots$ converges conditionally, converges absolutely, or diverges. *[2]*

SOLUTION. This series converges by the Alternating Series Test:

First, since $\dfrac{\ln(n)}{n} \geq 0$ for all $n \geq 1$, and is > 0 for all $n \geq 2$, the $(-1)^{n+1}$ in the term $\dfrac{(-1)^{n+1}\ln(n)}{n}$ causes the series to alternate between positive and negative.

Second, $\dfrac{d}{dx}\left(\dfrac{\ln(x)}{x}\right) = \dfrac{\left[\frac{d}{dx}\ln(x)\right]\cdot x - \ln(x)\cdot\left[\frac{d}{dx}x\right]}{x^2} = \dfrac{\frac{1}{x}\cdot x - \ln(x)\cdot 1}{x^2} = \dfrac{1-\ln(x)}{x^2}$, so $\dfrac{\ln(x)}{x}$ is decreasing as soon as $\ln(x) > 1$, which happens as soon as $x > e$. It follows that $\left|\dfrac{(-1)^{n+1}\ln(n)}{n}\right| = \dfrac{\ln(n)}{n}$ is (strictly) decreasing once $n > e$, *i.e.* for all $n \geq 3$.

Third, with a little help from l'Hôpital's Rule, we have

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\ln(n)}{n}\right| = \lim_{n\to\infty}\frac{\ln(n)}{n} = \lim_{x\to\infty}\frac{\ln(x)}{x}\begin{matrix}\to\infty\\ \to\infty\end{matrix} = \lim_{x\to\infty}\frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}x}$$
$$= \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = \lim_{x\to\infty}\frac{1}{x} = 0\,.$$

Since all three hypotheses of the Alternating Series Test are satisfied by the series, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\ln(n)}{n}$ converges.

This convergence is conditional because the corresponding series of positive terms, $\displaystyle\sum_{n=1}^{\infty}\frac{\ln(n)}{n}$ diverges by the Basic Comaprison Test: for each $n \geq 3$, we have $\ln(n) > \ln(e) = 1$, so $\dfrac{\ln(n)}{n} > \dfrac{1}{n}$, and the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. (Shown in class, and easily checked using the p-Test ...) ■

A *power series* is a series of the form $\sum_{n=0}^{\infty} a_n x^n$, where x is a variable and each a_n is a constant. It's basically a polynomial of infinite degree, one major difference being that while a polynomial of finite degree will be defined for all x, there is no guarantee that a power series will converge for all x. Indeed, there is no guarantee that it will converge for any x other than 0:

2. Show that the power series $\sum_{n=0}^{\infty} n!x^n$ converges only when $x = 0$. *[1]*

NOTE: If k is a positive integer, then $k!$ ("k factorial") denotes the product of all the positive integers less than or equal to k, *i.e.* $k! = k(k-1)(k-2)\cdots 3\cdot 2\cdot 1$. To make all sorts of formulas work nicely when $k = 0$, we arbitrarily define $0! = 1$.

HINT: $k!$ grows very fast with k, eventually overtaking and then running away from a^k for any $a \in \mathbb{R}$. You may use this fact without further ado.

SOLUTION. First, when $x = 0$, every term $n!x^n$ of the series becomes 0 except, ironically, the term for $n = 0$: $0!0^0 = 1\cdot 1 = 1$, because 0^0 also gets arbitrarily defined to be equal to 1 to get various formulas to work nicely. As $1 + 0 + 0 + 0 + \cdots$ obviously converges to 1, since that is what every partial sum is equal to, the given power series converges when $x = 0$.

Second, suppose $x \neq 0$. If $|x| \geq 1$, then $\lim_{n\to\infty} |n!x^n| = \lim_{n\to\infty} n!|x|^n = \infty \neq 0$, since $n! \to \infty$ and $|x|^n \geq 1$ as $n \to \infty$. It follows that $\lim_{n\to\infty} n!x^n \neq 0$ when $|x| \geq 1$. On the other hand, if $0 < |x| < 1$, we can still use the fact that $n!$ grows faster than any exponential, from which it follows that $\lim_{n\to\infty} |n!x^n| = \lim_{n\to\infty} \frac{n!}{\left(\frac{1}{|x|}\right)} = \infty$, and hence that $\lim_{n\to\infty} n!x^n \neq 0$. Either way, if $x \neq 0$, the given series diverges by the Divergence Test.

Thus the power series $\sum_{n=0}^{\infty} n!x^n$ converges only when $x = 0$. ■

3. Express $\frac{1}{1+x}$ as a power series. For which values of x does the series converge? *[2]*

SOLUTION. Running the formula for the sum of an infinite geometric series, $a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$, in reverse gives us:

$$\begin{aligned}\frac{1}{1+x} = \frac{1}{1-(-x)} &= 1 + (-x) + (-x)^2 + (-x)^3 + \cdots \\ &= 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n\end{aligned}$$

A geometric series converges exactly when $a = 0$ or when $|r| < 1$. As the first term in our series is $a = 1 \neq 0$, it follows that the series obtained above converges exactly when $|r| = |-x| = |x| < 1$, that is, for $-1 < x < 1$, and diverges otherwise. ■

4. Express $\ln(1+x)$ as a power series. For which values of x does it converge? *[2]*

HINT: How is $\ln(1+x)$ related to $\dfrac{1}{1+x}$?

SOLUTION. Following the hint, observe that, with the help of the substitution $u = 1+t$, so $du = dt$ and $\begin{array}{ccc} t & 0 & x \\ u & 1 & 1+x \end{array}$,

$$\int_0^x \frac{1}{1+t}\,dt = \int_1^{1+x} \frac{1}{u}\,du = \ln(u)\Big|_1^{1+x} = \ln(1+x) - \ln(1) = \ln(1+x) - 0 = \ln(1+x)\,.$$

Since we know from answering question **3** above that $\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$, this suggests the following cheap trick:

$$\begin{aligned}
\ln(1+x) &= \int_0^x \frac{1}{1+t}\,dt = \int_0^x \left(1 - t + t^2 - t^3 + \cdots\right)\,dt \\
&= \int_0^x 1\,dt - \int_0^x t\,dt + \int_0^x t^2\,dt - \int_0^x t^3\,dt + \cdots \\
&= t\Big|_0^x - \frac{t^2}{2}\Big|_0^t + \frac{t^3}{3}\Big|_0^t - \frac{t^4}{4}\Big|_0^t + \cdots \\
&= [x - 0] - \left[\frac{x^2}{2} - \frac{0^2}{2}\right] + \left[\frac{x^3}{3} - \frac{0^3}{3}\right] - \left[\frac{x^4}{4} - \frac{0^4}{4}\right] + \cdots \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}
\end{aligned}$$

The only dubious part here is the bit where the integral of the series becomes the series of integrals, but this turns out to be safe to do when the power series converges absolutely, which any geometric series does when it converges at all.

It turns out that the series we have obtained for $\ln(1+x)$ converges at all the points its parent series did, plus at one point, namely $x = 1$, that its parent series diverges at.

First, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$ converges absolutely when $|x| < 1$ by the Comparison Test, since for each $n \geq 0$, $\left|\dfrac{(-1)^n x^{n+1}}{n+1}\right| = \dfrac{|x|^{n+1}}{n+1} \leq |x|^{n+1}$, and the geometric series $\displaystyle\sum_{n=0}^{\infty} |x|^{n+1} = 1 + |x| + |x|^2 + \cdots$ converges when $|r| = |x| < 1$.

Second, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$ diverges when $|x| > 1$ by the Divergence Test. Note that $\displaystyle\lim_{n\to\infty} \frac{|x|^{n+1}}{n+1} = \infty \neq 0$ because the exponential growth of $|x|^{n+1}$ when $|x| > 1$ beats the linear growth of $n+1$. (How could you check this using what we know about limits?) It follows that $\lim_{n\to\infty} \frac{(-1)^n x^{n+1}}{n+1} \neq 0$, so the corresponding series fails the Divergence Test.

Third, we need to check what happens when $|x| = 1$, *i.e.* when $x = -1$ and when $x = 1$. Plugging $x = -1$ into the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$ gives us

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{-1}{n+1} = -\sum_{n=0} \frac{1}{n+1} = -\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right],$$

namely the negative of the harmonic series. We know from class that the harmonic series diverges, and if we didn't, we could easily check that it diverges using the p-Test.

On the other hand, plugging in $x = 1$ into $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$ gives us

$$\sum_{n=0}^{\infty} \frac{(-1)^n 1^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$

namely the alternating harmonic series,. We know from class that this converges, albeit conditionally, from class, and if we didn't, we could easily check that it converges using the Alternating Series Test.

Thus the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$ converges for $-1 < x \leq 1$ and diverges otherwise. ■

5. Use your answer to **4** to find the sum of the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, then check your answer by using `SageMath`. *[2]*

HINT: `SageMath` has a `sum` command, which we saw way back in Assignment #1. It can be used to sum infinite series, too.

SOLUTION. As noted in the solution to **3** above, the alternating harmonic series, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, is what we get when we plug $x = 1$ into the power series, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$, that is equal to $\ln(1 + x)$ when it converges, which it does for $x = 1$. It follows that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln(1 + 1) = \ln(2)$.

We check this by using `SageMath`'s `sum` command, which we used way back in Assignment #1 to compute Left-Hand Rule sums:

```
sage:  var("n")
n
sage:  sum((-1)^n/(n+1),n,0,oo)
log(2)
```

Since `SageMath` calls the natural logarithm function `log`, this confirms our calculation of the sum of the alternating harmonic series. Note `SageMath`'s use of two lower case o's, *i.e.* `oo`, for ∞. ■

6. Use `SageMath` to find the sums of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$. *[1]*

Solution. Continuing the session begun in checking the answer to **5** above:

```
sage:  sum(1/n^2,n,1,oo)
1/6*pi^2
sage:  sum((-1)^(n+1)/n^2,n,1,oo)
1/12*pi^2
```

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$ according to `SageMath`.

In case you're curious, showing that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ is almost within reach of first-year calculus. It require developing some unusual trigonometric identities – this is within reach – and one fact about interchanging limits and infinite sums, a version of something called Tannery's Theorem whose proof is beyond the scope of a first-year calculus course. Some algebraic trickery is all you need to show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$, given that you know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. ■