Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Winter 2022

Assignment #5
A separable differential equation
Due on Friday, 18 February. (May be submitted on paper or via Blackboard.)*

Please show all your work. As with all the assignments in this course, unless stated otherwise on the assignment, you are permitted to work together and look things up, so long as you acknowledge the sources you used and the people you worked with.

One of the big applications of calculus, usually studied in second- or third-year in a separate course or courses†, is in solving equations where derivatives of unknown functions appear in order to find those functions.

For a cheap example, consider the equation $\frac{dy}{dx} = x$. It's not too hard to see that a general solution would be something like $y = \int x\,dx = \frac{x^2}{2} + C$, and you could determine C with additional information such as knowing the value of y for some particular x. (*e.g.* if $y = 1$ when $x = 1$, you would have to have that $C = y - \frac{x^2}{2} = 1 - \frac{1^2}{2} = \frac{1}{2}$.)

For a more expensive example, consider the equation $\frac{dy}{dx} = y$, with the extra information or "initial condition" that $y = 3$ when $x = 0$. This one can be solved with a cheap trick that is hard to justify theoretically, but which works: move everything involving y to one side and everything involving x to the other side of the equation, and then integrate each side with respect to the variable on that side:

$$\frac{dy}{dx} = y \implies \frac{1}{y}\cdot\frac{dy}{dx} = 1 \implies \frac{dy}{y} = 1\,dx \implies \int \frac{dy}{y} = \int 1\,dx$$
$$\implies \ln(y) = x + C \implies y = e^{x+C} = e^C e^x = Ke^x$$

When $x = 0$, $y = Ke^0 = K \cdot 1 = K$ and, since we are supposed to have that $y = 3$ when $x = 0$, it follows that $K = 3$. Thus the given differential equation with the given initial condition has solution $y = 3e^x$.

1. Solve the differential equation $\frac{dy}{dx} = \frac{\sin(x)}{\sin(y)}$, subject to the initial condition that $y = \frac{\pi}{2}$ when $x = 0$. *[5]*

Solution. We will use the same basic method, called "separation of variables", as in the

* All else failing, please email your solutions to the instructor at: `sbilaniuk@trentu.ca`

† At Trent these are MATH-PHYS 2150H *Ordinary Differential Equations* and MATH-PHYS 3150H *Partial Differential Equations*.

second example above.

$$\begin{aligned}\frac{dy}{dx} = \frac{\sin(x)}{\sin(y)} &\implies \sin(y)\frac{dy}{dx} = \sin(x) \implies \sin(y)\,dy = \sin(x)\,dx \\ &\implies \int \sin(y)\,dy = \int \sin(x)\,dx \implies -\cos(y) = -\cos(x) + C \\ &\implies \cos(y) = \cos(x) - C \implies y = \arccos\left(\cos(x) - C\right)\end{aligned}$$

It remains to determine the value of the constant C. Since we are given the initial condition $y = \frac{\pi}{2}$ when $x = 0$, we must have:

$$\frac{\pi}{2} = \arccos\left(\cos(0) - C\right) = \arccos(1 - C) \implies 0 = \cos\left(\frac{\pi}{2}\right) = 1 - C \implies C = 1$$

It follows that $y = \arccos\left(\cos(x) - 1\right)$. ■

2. Use `SageMath` to solve the differential equation from question **1**, with the same initial condition. *[5]*

NOTE: You will want to look up the `desolve` command, described in §4.22 (pp. 192 on) of *Sage for Undergraduates*, as well as the `function` and `diff` operators. Please note that there is an error in how the `function` operator is described: where the book tells you should type `y = function('y',x)`, you should type `y = function('y')(x)` instead. (The format of the function was probably changed after that part of the book was written.)

SOLUTION. Here we go:

```
[1]: y = function('y')(x)
     desolve( diff(y,x) == sin(x)/sin(y), y, ics=[0,pi/2])

[1]: -cos(y(x)) == -cos(x) + 1

[2]: solve( -cos(y(x)) == -cos(x) + 1, y)

[2]: [y(x) == arccos(cos(x) - 1)]
```

`SageMath` has done everything here immediately except for solving for y in the final answer, possibly because arccos inverts only the part of $\cos(x)$ where $0 \le x \le \pi$, so the equation $-\cos(y) = -\cos(x) + 1$ is arguably more general. Using the `solve` command lets one solve for y explicitly and get the same answer we obtained in the solution to question **1**. ■