Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2022

Solutions to Assignment #11
A Power Series For $\sqrt{1+x}$
Due on Friday, 8 April.

Please show all your work. As with all the assignments in this course, unless stated otherwise on the assignment, you are permitted to work together and look things up, so long as you acknowledge the sources you used and the people you worked with.

If $r \in \mathbb{R}$ and $k \geq 1$ is an integer, then the *binomial of r and k* is

$$\binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!} \,.$$

Thus $\binom{r}{1} = r$, $\binom{r}{2} = \frac{r(r-1)}{2}$, $\binom{r}{3} = \frac{r(r-1)(r-2)}{6}$, and so on. To make various formulas work nicely, we let $\binom{r}{0} = 1$. Note that when r is a positive integer, this coincides with the usual definition of binomial coefficients. *Newton's Binomial Theorem* extends the usual Binomial Theorem for expanding expressions like $(x+y)^n$ (for integer $n \geq 1$) and states that if r, x, and a are real numbers with $|x| < |a|$, then

$$\begin{aligned}(a+x)^r &= \sum_{n=0}^{\infty} \binom{r}{n} a^{r-n} x^n \\ &= a^r + ra^{r-1}x + \frac{r(r-1)}{2}a^{r-2}x^2 + \frac{r(r-1)(r-2)}{6}a^{r-3}x^3 + \cdots \,.\end{aligned}$$

1. Suppose $|x| < 1$. We can expand $\sqrt{1+x} = (1+x)^{1/2}$ as a power series using Newton's Binomial Theorem. Find the radius and interval of convergence of this series. *[10]*

SOLUTION. If we expand $\sqrt{1+x} = (1+x)^{1/2}$ as a power series using Newton's Binomial Theorem, with $a = 1$ and $r = \frac{1}{2}$, we get:

$$\begin{aligned}(1+x)^{1/2} &= \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} 1^{\frac{1}{2}-n} x^n = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n \\ &= 1 + \frac{1}{2} \cdot x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2} \cdot x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{6} \cdot x^3 + \cdots\end{aligned}$$

This is the series whose radius and interval of convergence we need to find.

As usual, we first try the Ratio Test to determine the radius of convergence:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\binom{\frac{1}{2}}{n+1}x^{n+1}}{\binom{\frac{1}{2}}{n}x^n}\right| = \lim_{n\to\infty}\left|\frac{\binom{\frac{1}{2}}{n+1}}{\binom{\frac{1}{2}}{n}}\cdot x\right| = \lim_{n\to\infty}\left|\frac{\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-(n+1)+1\right)}{(n+1)!}}{\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!}}\cdot x\right|$$

$$= \lim_{n\to\infty}\left|\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-(n+1)+1\right)}{(n+1)!}\cdot\frac{n!}{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}\cdot x\right|$$

$$= \lim_{n\to\infty}\left|\frac{\frac{1}{2}-(n+1)+1}{n+1}\cdot x\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{2}-n}{n+1}\cdot x\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{2}-n}{n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}}\cdot x\right|$$

$$= \lim_{n\to\infty}\left|\frac{\frac{1}{2n}-1}{1+\frac{1}{n}}\cdot x\right| = \left|\frac{0-1}{1+0}\cdot x\right| = |(-1)\cdot x| = |x|$$

By the Ratio Test, it follows that the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$, so the radius of convergence is 1.

It remains to determine whether the series converges when $|x| = 1$, that is, when $x = -1$ and when $x = 1$. This, sadly, is pretty hard to do from scratch with the tools acquired in this course, so the easiest thing to do is to look it up. (Just as a reminder, unless otherwise specified on an assignment, you are allowed to collaborate and look things up on these assignments, so long as you acknowledge who you worked with and provide references for any sources you used.) One could, for example, just use a search engine to search for, say, "Newton's binomial theorem".

If one does so, one or more of the results that are likely to turn up are Wikipedia articles. (Wikipedia is often a pretty good resource for mathematics.) In this case, the article *Binomial theorem* linked to the article *Binomial Series*, at

`https://en.wikipedia.org/wiki/Binomial_series`

that actually has an outline of a proof of when the series above converges. (Go look at it! :-) Granted, it covers a lot more than is done here – in particular, the numbers we are calling r and x here could be complex numbers – but you can pick out what happens when $r = \frac{1}{2}$ and $x = \pm 1$. It turns out that the series converges in these cases, so the interval of convergence is $[-1, 1]$. ■