Taylor Series III

The remainder term of a Taylor series

$f(x)$ The taylor series at a is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

$$T_n(x) = \sum_{n=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots$$

is the taylor polynomial at "a" of $f(x)$ degree n

the n^{th} remainder term in this situation $\quad R_n(x) = f(x) - T_n(x)$

Fact: In this situation, $R_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}$

for some "t" between "a" and "x" (strictly in between)

Cheap example: $\quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x

$= \lim_{n\to\infty} T_n(x)$ for all x

$$\lim_{n\to\infty} \underbrace{[e^x - T_n(x)]}_{R_n(x)} = 0$$

So it's enough to show that

$$\lim_{n\to\infty} R_n(x) = 0$$

but $\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{\frac{d^{n+1}}{dt^{n+1}} e^x \big|_{x=t}}{n!}(x-0)^{n+1}$

$$= \lim_{n\to\infty} \frac{e^x}{(n+1)!} x^{n+1}$$

if $x>0 \quad 0<t<x$

$\quad x<0 \quad 0>t>x$

$$0 < e^t < e^{|t|} < e^x$$

since e^x is a increasing function

$$0 < \left|\frac{e^t}{(n+1)!} x^{n+1}\right| < \left|\frac{e^{|x|}}{(n+1)!} x^{n+1}\right| \to 0 \quad \text{as } n\to\infty$$

so $\lim_{n\to\infty} \frac{e^t}{(n+1)!} x^{n+1} = 0$ is $\lim_{n\to\infty} R_n(x) = 0$

We'll use this technology to show that e is irrational

Suppose, for the sake of arrgument the $e = \frac{a}{b}$ for some positive integers a&b

[Assume "e" is achally rational]

$$e = e^1 = \sum_{n=0}^{\infty} \frac{1^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$= T_n(1) + R_n(1) \text{ for all } n$$

$$= \left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}\right) + \frac{e^t}{(n+1)!} \cdot \cancel{1^n} \qquad 0 < t < 1$$

but $e^t < e^1 < 3$ (as e^x is 1 increasing)

so $0 < R_n(1) < \frac{3}{(n+1)!}$

pick $n > 3b$. Then
$n \geq 3$

$$\frac{a}{b} = e = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}\right) + \frac{e^t}{(n+1)!}$$

then

$$0 < \frac{a}{b} - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}\right) < R_n(1) < \frac{3}{(n+1)!}$$

$n \geq 3$
$n+1 \geq 4$

so

$$0 = 0n! < \frac{n!a}{b} - \left(n! + \frac{n!}{1!} + \frac{n!}{2!} + \dots + \frac{n!}{n!}\right) < n!R_n(1) < \frac{3n!}{(n+1)!} \to \frac{3}{n+1} \leq \frac{3}{4} < 1$$

integer since b divids n!

So this means that $\frac{n!a}{b} - \left(n! + \frac{n!}{1!} + \frac{n!}{2!} + \dots + \frac{n!}{n!}\right)$ is an integer between 0 & 1,

which can't happen.

Thus the assumtion that we could write $e = \frac{a}{b}$ must be wrong, is e is irrational