Taylor Series II

More examples

ex: $f(x) = x + 0$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$x+0$	$0+0=0$
1	1	1
2	0	0
3	0	0
4	0	0

So the taylor series of $f(x)$ at 0 is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$$\sum_{n=0}^{\infty} \cancel{\frac{0}{0!}} x^0 + \frac{1}{1!} x^1 + \cancel{\frac{0}{2!}} x^2 + \cancel{\frac{0}{3!}} x^3 \ldots$$

$$= x$$

General fact:

if $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial it's Taylor series at 0 is it self

Even more general if $f(x) = \sum_{n=0}^{\infty} a_r x^n$ then the series is it's Taylor series

ex: $f(x) = \arctan(x)$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\arctan(x)$	0
1	$\frac{1}{1+x^2}$	1
2	$\frac{-2x}{(1+x^2)^2}$	0
3	$\frac{-2+2x^2}{(1+x^2)^3}$	-2
4	??	?

& now, without trying to use taylor's formula

$$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

sum of a geometric series with $a=1$ & $r=-x^2$

$$= 1 - x^2 + x^4 - x^6 + x^8 - \ldots$$

so...

$$\arctan(x) = \int \frac{1}{1+t^2}\,dt$$

$$= \int (1-x^2+x^4-x^6+x^8 \ldots)\,dt$$

$$= C+t-\frac{t^3}{3}+\frac{t^5}{5}-\frac{t^7}{7}+\ldots \qquad t=x$$

$$= C+\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

what is C? plug in $x=0$ to both sides

$$\underbrace{\arctan(0)}_{=0} = C+\cancel{\sum_{n=0}^{\infty} \frac{(-1)^n 0^{2n+1}}{2n+1}}^{=0}$$

$$0=C$$

Thus the taylor series of $\arctan(x)$ at 0 is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

What are the radius & interval of convergence of this series?

If you differentiate or integrate a power series term by term the radius of convergence changes not. Convergence may change at the endpoint of the integral of convergence

$$\frac{1}{1+x^2} = 1-x^2+x^4-x^6$$

is a geometric series with $|r| = -x^2 = x^2 < 1$
so it converges exactly when $|x|<1$

ie $R=1$ & the interval of convergence is $(-1,1)$

It follow that our series for $\arctan(x)$ has $R=1$

At the endpoints?

$x=1$ & $x=-1$

at $x=-1$ $\quad \sum_{n=0}^{\infty} \frac{\overbrace{(-1)^n(-1)^{2n+1}}^{=(-1)^{3n+1}}}{2n+1} = \frac{-1}{1}+\frac{1}{3}+\frac{-1}{5}+\frac{1}{7}\ldots$

This converges by the Alternating series test (you check the details)

The convergence is conditional because $\sum_{n=0}^{\infty} \frac{1}{2n+1}$, diverges by the p-test

at $x=1$: $\sum_{n=0}^{\infty} \frac{(-1)^n (1)^{2n+1}}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots$

This converges by the Alternating series test

The convergence is conditional because $\sum_{n=0}^{\infty} \frac{1}{2n+1}$, diverges by the p-test

So $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots = \arctan(1)$

$= \pi/4$