Taylor's Series I

Power Series (The short form) [Virtual cookies if you get the movie reference]

A series of the form $\sum_{n=0}^{\infty} a_n x^n$ or $\sum_{n=0}^{\infty} a_n (x-c)^n$

such a series has a radius of convergence $0 \le R \le \infty$

- If $R=0$, the series converges only at $x=0$ (or $x=c$)
- Otherwise, it converges absolutly for $|x|<R$ (or $|x-c|<R$) & diverges for any $|x| \geq R$ (or $|x-c|>R$)

at $x = \pm R$ the series may converge or may diverge

within the radius of convergence you can differentiate & integrate term by term (& not change the radius of curvergent)

If $\sum_{n=0}^{\infty} a_n (x-c)^n$ is a power series and we think of it as a function $f(x)$

then for each $n \geq 0$

$a_n = \frac{f^{(n)}(c)}{n!}$ n^{th} derivative of $f(x)$

$f^{0}(x) = f(x)$

Thus Taylor's Formula:

Given $f(x)$, <u>if</u> it can be expanded as a power series around $x=c$, the series is $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$

WARNING: the series might not converge to $f(x)$ (except at $x=c$). Examples of this are rare.

Example: Find the taylor series of $\cos(x)$ at 0

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos(x)$	1
1	$-\sin(x)$	0
2	$-\cos(x)$	-1
3	$\sin(x)$	0
4	$\cos(x)$	1

So $f^{(n)}(0) = \begin{cases} (-1)^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd (ie } n=2K+1) \end{cases}$

So the series is

$$\frac{1}{0!}x^0 + \frac{-1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{-1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^K}{(2K)!} x^{2K}$$

$R = ?$

Ratio test

$$= \lim_{n\to 0} \left| \frac{\frac{(-1)^{K+1} x^{2K+2}}{(2K+2)!}}{\frac{(-1)^K x^{2K}}{(2K)!}} \right|$$

$$= \lim_{n\to 0} \left| \frac{\cancel{(-1)^{K+1}} x^{\cancel{2K+}2}}{(2K+2)!} \cdot \frac{(2K)!}{\cancel{(-1)^K} \cancel{x^{2K}}} \right|$$

$$(2K+2)(2K+1)\underbrace{(2K)(2K-1)\ldots}_{(2K)}$$

$$= \lim_{n\to 0} \left| \frac{x^2}{(2K+2)(2K+1)} \right| \quad \begin{matrix} \to x^2 \\ \to \infty \end{matrix}$$

$= \lim_{n\to 0} = 0$ So by the Ratio test, the series converges for all $|x|<\infty$ (ie all x)

ie $R = \infty$

If L is the limit coming from the ratio test, then you get R by solving for when $L<1$.

Without doing all this again, how do we get a power series for $\sin(x)$?

$$-\sin(x) = \frac{d}{dx}\cos(x)$$

$$\sin(x) = (-1)\frac{d}{dx}\cos(x)$$

$$= (-1)\frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{(-1)^K x^{2K}}{(2K)!}\right)$$

$$= (-1)\sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{(-1)^K x^{2K}}{(2K)!}\right)$$

$$= (-1) \sum_{n=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{d}{dx} x^{2k}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{k+1}}{(2k)!} 2k\, x^{2k-1}$$

since $\frac{d}{dx}(x^{2\cdot 0}) = 0$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)!} x^{2k-1}$$

$(2k)!$

$(2k)(2k-1)(2k-2)\ldots$

$(2k)(2k-1)!$

If we set $k = n+1$ ie $n = k-1$, then this series looks like

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+2}}{(2n+1)!} x^{2n+1}$$

usual series at 0 for $\sin(x)$.