Power Series

A series of the form $\sum_{n=0}^{\infty} c_n x^n$, where x is a variable

Idea: Rewrite functions of x as power series to make them easier to handle

ex: $\int e^x \, dx$ has no nice antiderivative but you can write one as a power series

Prototype: Geometric Series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

$a=1$, $r=x$ which cvgs when $|r|=|x|<1$

For which value of x dose $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges?

Ratio Test: $\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right|$

$$= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{n+1} \right|$$

$$= \lim_{n\to\infty} \frac{|x|}{\underbrace{n+1}_{\to\infty}} = 0 < 1$$

so the series converges absolutly (for all x)

This series sums to e^x

$\boxed{\text{Note } e^0 = 1}$

How can you tell?

$$\sum_{n=0}^{\infty} \frac{0^n}{n!} = \overbrace{\frac{0^0}{0!}}^{=1} + \overbrace{\cancel{\frac{0^1}{1!}}}^{=0} + \overbrace{\cancel{\frac{0^2}{2!}}}^{=0} + \cdots = \frac{1}{1} = e^0$$

Is $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$?

They both satisfy the differential equation $\frac{dy}{dx} = y$ with initial condition that $y=1$ when $x=0$

If $y = e^x$ then $\frac{dy}{dx} = \frac{d}{dx} e^x = e^x = y$ and $e^0 = 1$

If $y = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ then $\frac{dy}{dx} =$

$$= \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)$$

$$= \frac{d}{dx}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)$$

$$= 0 + 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$\frac{d}{dx}\left(\frac{x^n}{n!}\right) = \frac{nx^{n-1}}{n!} = \frac{nx^{n-1}}{(n-1)!}$$

$\hookrightarrow \sum_{k=0}^{\infty} \frac{x^k}{k!} = y \quad (k = n-1)$

and $y=1$ when $x=0$

WARNING:

Different functions could have the same power series

and so the power series is only equal to <u>one</u> of them

ex:

$$f(x) \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

has (via taylor's formula) a power series expansion of

$0+0+0+0+\ldots = \sum_{n=0}^{\infty} 0^{n+1}$ which is the same as the expansion of $g(x) = 0$

<u>A bit about Taylor's series</u>

If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ (when it converges then)

$$f(0) = \sum_{n=0}^{\infty} c_n 0^n = c_0 0^0 \cancel{+ c_1 0^1 + \cdots} = c_0(1) = c_0$$

$$f'(0) = \frac{d}{dx}\left(c_0 + c_1 x + c_2 x^2 + \cdots\right)\Big|_{x=0}$$

$$= \left(0 + c_1 + 2c_2 x + 3c_3 x^2 + \cdots\right)\Big|_{x=0}$$

$$= c_1$$

$$f''(0) = \frac{d}{dx}\left(0 + c_1 + 2c_2 x + 3c_3 x^2 + \cdots\right)\Big|_{x=0}$$

$$= \cdots$$

$$= c_2$$

$$f^n(0) = c_n$$

↳ nth derivative at $x=0$

$f(x) = \sum_{n=0}^{\infty} c_n x^n$ then $c_n = \dfrac{f^{(n)}(0)}{n!}$

Taylor's formula : If $f(x)$ is infinitely differentiable at $x=0$ then it's Taylor series if $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$