Series VIII

And now, the Root test

Given a series $\sum_{n=0}^{\infty} a_n$, this time we cumpute

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \begin{cases} <1 & \text{the series converges absolutly} \\ =1 & \text{no information} \\ >1 & \text{the series diverges} \end{cases}$$

ex: When does $\sum_{n=0}^{\infty} \frac{x^n}{n^n}$ converges? (ie for which x)

$\boxed{\text{Note : } 0^0 = 1}$

$$\lim_{n\to\infty} \left|\frac{x^n}{n^n}\right|^{1/n} = \lim_{n\to\infty} \left|\frac{(x^n)^{1/n}}{(n^n)^{1/n}}\right| = \lim_{n\to\infty} \left|\frac{x}{n}\right| = \lim_{n\to\infty} \frac{|x|}{n} = 0 < 1$$

↳ So the series converges for all x

ex: For which x does the series $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$ converges?

$n!$ ("n factorial") means (for ≥ 1)
$= n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$
so $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$

Stirlings Formula → don't need to know

$n!$ act more or less preportionaly to $\left(\frac{n}{e}\right)^n \sqrt{n}$ when n is large

→ Ratio Test (cause not everything is an n^{th} power)

$$\lim_{n\to\infty} \left|\frac{\frac{3^{n+1} x^{n+1}}{(n+1)!}}{\frac{3^n x^n}{n!}}\right| = \lim_{n\to\infty} \left|\frac{3^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n x^n}\right| = \lim_{n\to\infty} \left|\frac{3x}{n+1}\right| = 3|x| \lim_{n\to\infty} \frac{1}{n+1} = 3|x| \cdot 0 < 1$$

Thus the series converges absolutly for all x

(Infact, $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!} = e^{3x}$)

ex: How about checking when $\sum_{n=0}^{\infty} \frac{x^{2n+2} (-1)^{3n}}{(2n)!}$ converges??

Ratio test (cause not everything is an n^{th} power) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$\lim_{n\to\infty}\left|\frac{\frac{x^{2(n+1)+2}(-1)^{3(n+1)}}{(2(n+1))!}}{\frac{x^{2n+2}(-1)^{3n}}{(2n)!}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+4}(-1)^{3n+3}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n+2}(-1)^{3n}}\right| = \lim_{n\to\infty}\left|\frac{x^2(-1)^3}{(2n+2)(2n+1)}\right|$$

$$= x^2\lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} = 0 < 1$$ so the sereis converges for all x by the Ratio test

ex: When dose $\sum_{n=0}^{\infty}\frac{3^n x^{n+3}}{4^n} = \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n x^{n+3}$ converges?

<u>Root test</u>

A bit easier to use if we rewrite the series

$$\sum_{n=0}^{\infty}\frac{3^n x^{n+3}}{4^n} = x^3\sum_{n=0}^{\infty}\frac{3^n x^n}{4^n}$$

$$\lim_{n\to\infty}\left|\frac{3^n x^n}{4^n}\right|^{1/n} = \lim_{n\to\infty}\left|\frac{3x}{4}\right| = \frac{3}{4}|x|$$

If we don't rewrite it

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left|\frac{3^n x^{n+3}}{4^n}\right|^{1/n}$$

$$= \lim_{n\to\infty}\underbrace{\left|\frac{3x}{4}\right|}_{\to\frac{3}{4}|x|}\cdot\underbrace{|x^3|^{1/n}}_{\to 1}$$

So by the Root test, this series converges when

$\frac{3}{4}|x| < 1$ (ie $|x| < \frac{4}{3}$)

and diverges when $\frac{3}{4}|x| > 1$ (ie $|x| > \frac{4}{3}$)

so what happens when $\frac{3}{4}|x| = 1$ ie: when $x = -\frac{4}{3}$ or $x = \frac{4}{3}$

$x = \frac{4}{3}$: $\sum_{n=0}^{\infty}\frac{3^n}{4^n}\cdot\left(\frac{4}{3}\right)^{n+3}$

$\sum_{n=0}^{\infty}\cancel{\frac{3^n}{4^n}\cdot\frac{4^n}{3^n}}\cdot\frac{4^3}{3^3} = \sum_{n=0}^{\infty}\frac{64}{27}$ which diverges by the divergence test because $\lim_{n\to\infty}\frac{64}{27} \neq 0$

$x = -\frac{4}{3}$: $\sum_{n=0}^{\infty}\frac{3^n}{4^n}\cdot\left(\frac{4}{3}\right)^{n+3}$

$\sum_{n=0}^{\infty}\frac{3^n}{4^n}\cdot(-1)^{n+3}\cdot\left(\frac{4}{3}\right)^{n+3}$

$\sum_{n=0}^{\infty}\cancel{\frac{3^n}{4^n}\cdot\frac{4^n}{3^n}}\cdot\frac{4^3}{3^3}\cdot(-1)^{n+3} = \sum_{n=0}^{\infty}(-1)^{n+3}\cdot\frac{64}{27}$ which diverges by the divergence test because $\lim_{n\to\infty}(-1)^{n+3}\cdot\frac{64}{27} \neq 0$