Lecture 18 Mar 18^{th}, 2022

Alternating Series Test:

Consider the series $\sum_{n=0}^{\infty} a_n$. Then if each $a_n \neq 0$ and

1) $a_{n+1} < 0 \iff a_n > 0$ and,
2) $|a_{n+1}| \leq |a_n|$ and,
3) $\lim_{n\to\infty} |a_n| = 0$,

then the series converges.

ex/ $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

$a_n = \frac{(-1)^n}{n+1} \neq 0$ for all n.

(1) if $a_{n+1} = \frac{(-1)^{n+1}}{n+2} < 0$, then $n+1$ is odd, so n is even, and $a_n = \frac{(-1)^n}{n+1} > 0$. This reasoning is reversable.
$\therefore a_{n+1} < 0 \iff a_n > 0$.

(2) $|a_{n+1}| = \frac{1}{n+2} < \frac{1}{n+1} = |a_n|$
$\therefore |a_{n+1}| \leq |a_n|$.

(3) $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n+1} = 0$
$\therefore \lim_{n\to\infty} |a_n| = 0$

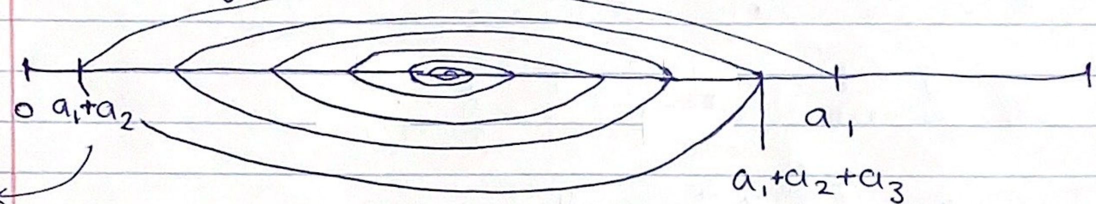
$\therefore$ this series converges by the alternating series test.
(even though $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges.

A series $\sum_{n=0}^{\infty} a_n$ converges <u>absolutely</u> if $\sum_{n=0}^{\infty} |a_n|$ converges.

It converges <u>conditionally</u> if $\sum_{n=0}^{\infty} a_n$ converges but $\sum_{n=0}^{\infty} |a_n|$ does not.

Why does the alternating series test work?

Suppose $\sum_{n=0}^{\infty} a_n$ and it satisfies conditions 1→3 of the alternating series test.



$a_2, a_4, a_6 \ldots$ are negative

Since $\lim_{n\to\infty} |a_n| = 0$, this hones in on some point.

ex/ $\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\ln(n^2)}$

1) This is an alternating series because $\ln(n^2) > 0$ if $n \geq 2$ and $\cos(n\pi) = \{1 \text{ if } n \text{ is even}, -1 \text{ if } n \text{ is odd}\}$.

2) Also, $|a_{n+1}| = \left|\frac{\cos((n+1)\pi)}{\ln((n+1)^2)}\right| = \frac{1}{\ln((n+1)^2)}$ and $|a_n| = \left|\frac{\cos(n\pi)}{\ln(n^2)}\right| = \frac{1}{\ln(n^2)}$.
Since $(n+1)^2 > n^2$, $\frac{1}{\ln((n+1)^2)} < \frac{1}{\ln(n^2)} \Rightarrow |a_{n+1}| < |a_n|$.

3) $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left|\frac{\cos(\pi n)}{\ln(n^2)}\right| = \lim_{n\to\infty} \frac{1}{\ln(n^2)} = 0$.

$\therefore$ this series converges.

Is this converging absolutely or conditionally?

Check if $\sum_{n=2}^{\infty} \left| \frac{\cos(n\pi)}{\ln(n^2)} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln(n^2)}$ converges.

$$\sum_{n=2}^{\infty} \frac{1}{\ln(n^2)} = \sum_{n=2}^{\infty} \frac{1}{2\ln(n)} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{\ln(n)}.$$

$\ln(n) < n$ for all $n \geq 1$

$\Rightarrow \frac{1}{\ln(n)} > \frac{1}{n}$ for all $n \geq 1$.

but $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by the p-test ($p = 1 \leq 1$).

so $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges by the Comparison test.

Since $\sum_{n=2}^{\infty} \left| \frac{\cos(n\pi)}{\ln(n^2)} \right|$ diverges,

$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\ln(n^2)}$ converges <u>conditionally</u>.