Series V

Dose $\sum_{k=2}^{\infty} \frac{1}{\ln(k)}$ converge or diverge

if it diverges we need

$\frac{1}{\ln(K)} > \frac{1}{K}$ (something that diverges)

$K > \ln(K)$ past some point $(K \geq 2)$

$\sum_{k=2}^{\infty} \frac{1}{K}$ diverges by the p test since $p = 1 \leq 1$

so $\sum_{K=2}^{\infty} \frac{1}{\ln(K)}$ diverges too by the comparison test

Ex: Dose $\sum_{n=0}^{\infty} \frac{\arctan(n)}{1+n^2}$ converges or divergens?

Using limit comparison test: What dominates?

$\frac{\arctan(n)}{1+n^2} < \frac{\pi/2}{n^2}$ since $1+n^2 > n^2$ for $n \geq 1$

but $\sum_{n=1}^{\infty} \frac{\pi/2}{n^2} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges by the p-test since $p=2>1$,

so $\sum_{n=0}^{\infty} \frac{\arctan(n)}{1+n^2}$ converges by the Comparison test

Using the limit comparison test $\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{\arctan(n)}{1+n^2}}$

$= \lim_{n\to\infty} \frac{n^2+1}{n^2 \arctan(n)}$

$= \lim_{n\to\infty} \frac{n^2+1}{n^2 \arctan(n)} \cdot \left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right)$

$$= \lim_{n\to\infty} \frac{1 + \overset{\to 0}{\frac{1}{n^2}}}{\underset{\to \frac{\pi}{2}}{\arctan(n)}} = \frac{1+0}{\pi/2}$$

$$= \frac{2}{\pi} > 0$$

So the limit comparison test says

$\sum_{n=0}^{\infty} \frac{\arctan(n)}{1+n^2}$ Converges if $\sum_{n=1}^{\infty} \frac{1}{n^2}$ dose

<u>Recall</u>: The Generalized P-test

$$\sum_{n=0}^{\infty} \frac{a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n + a_0}{b_\ell n^\ell \, b_{\ell-1} n^{\ell-1} + \ldots + b_1 n + b_0} \quad (\text{where } a_k \neq 0 \ \& \ b_\ell \neq 0)$$

Converges if $p = \ell - k > 1$ and diverges if $p = \ell - k \leq 1$

Proof: Using the P-test and the limit comparson test

We'll compare the given seires to $\frac{1}{n^p} = \frac{1}{n^{\ell-k}} = \frac{n^k}{n^\ell}$

$$\lim_{n\to\infty} \frac{\frac{a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n + a_0}{b_\ell n^\ell \, b_{\ell-1} n^{\ell-1} + \ldots + b_1 n + b_0}}{1/n^p}$$

$$= \lim_{n\to\infty} \frac{\overset{\Rightarrow n^{\ell-k}}{n^p}}{1} \cdot \frac{a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n + a_0}{b_\ell n^\ell \, b_{\ell-1} n^{\ell-1} + \ldots + b_1 n + b_0}$$

$$= \lim_{n\to\infty} \frac{a_k n^\ell + \ldots + a_1 n^{\ell-k+1} + a_0 n^{\ell-k}}{b_\ell n^\ell + \ldots + b_1 n + b_0} \cdot \frac{1/n^\ell}{1/n^\ell}$$

$$= \lim_{n\to\infty} \frac{a_k + \underbrace{\frac{a_{n-1}}{n} + \ldots + \frac{a_0}{n^k}}_{=0}}{b_\ell + \underbrace{\frac{b_{\ell-1}}{n} + \ldots + \frac{b_0}{n^\ell}}_{=0}} = \frac{a_k}{b_\ell} \neq 0$$

$\therefore \sum_{n=0}^{\infty} \frac{a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n + a_0}{b_\ell n^\ell \, b_{\ell-1} n^{\ell-1} + \ldots + b_1 n + b_0}$ Converges or diverges exactly as

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ does <u>ie</u> as $p = \ell - k > 1$ (converges) or

$p = \ell - k \leq 1$ (diverges)