Lecture 17 Mar. 15^{th}, 2022

ex/ Does $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$ converge or not?

Integral Test:

*by inspection, this function is positive and decreasing

$$\int_1^{\infty} \frac{1}{x^{\sqrt{2}}}\,dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x^{\sqrt{2}}}\,dx = \lim_{b\to\infty} \int_1^b x^{-\sqrt{2}}\,dx$$

$$= \lim_{b\to\infty} \frac{x^{-\sqrt{2}+1}}{-\sqrt{2}+1}\Bigg|_1^b = \frac{b^{-\sqrt{2}+1}}{-\sqrt{2}+1} - \frac{1^{-\sqrt{2}+1}}{-\sqrt{2}+1}$$

$$= \lim_{b\to\infty}\left[\frac{1}{1-\sqrt{2}} \cdot \cancelto{0}{\frac{1}{b^{\sqrt{2}-1}}} + \frac{1}{\sqrt{2}-1}\right]$$

$$= \frac{1}{\sqrt{2}-1}$$

$\therefore$ the series converges

This works because $\sqrt{2} \approx 1.4 > 1$.

So, <u>p-test</u>: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

Generalized p-test:

$\sum_{n=1}^{\infty} \frac{a_k n^k + \dots + a_1 n + a_0}{b_\ell n^\ell + \dots + b_1 n + b_0}$ converges if $p = \ell - k > 1$ and diverges if $p = \ell - k \le 1$.

Basic Comparison Test:

Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of positive terms. If $0 < a_n \le b_n$ past some point, then:

1) if $\sum_{n=0}^{\infty} b_n$ converges, so does $\sum_{n=0}^{\infty} a_n$

2) if $\sum_{n=0}^{\infty} a_n$ diverges, so does $\sum_{n=0}^{\infty} b_n$

ex/ Does $\sum_{n=0}^{\infty} \frac{1}{n^2+n+3}$ converge?

Note that $\frac{1}{n^2+n+3} < \frac{1}{n^2}$ for $n \ge 1$ because $n^2+n+3 > n^2$.

Since $\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges by the p-test (ie. $p=2>1$), it follows by the basic comparison test that:

$$\sum_{n=0}^{\infty} \frac{1}{n^2+n+3} \text{ converges too.}$$

ex/ Does $\sum_{n=0}^{\infty} \frac{1}{n^2-n+3}$ converge?

$\frac{1}{n^2-n+3} > \frac{1}{n^2}$ b/c $n^2-n+3 < n^2$ for $n > 3$.

What do we compare $\sum_{n=0}^{\infty} \frac{1}{n^2-n+3}$ to if we want to show it converges?

$\frac{1}{n^2-n+3} < \frac{2}{n^2} = \frac{1}{n^2/2}$ as past some point $\frac{n^2}{2} > n-3$ (namely $n \ge 3$) but then $n^2 - \frac{n^2}{2} < n^2-(n-3) = n^2-n+3$
$\Rightarrow \frac{n^2}{2} < n^2-n+3$.

We have $\frac{1}{n^2-n+3} < \frac{1}{n^2/2} = \frac{2}{n^2}$

Since $\sum_{n=0}^{\infty} \frac{2}{n^2} = 2\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges, so does $\sum_{n=0}^{\infty} \frac{1}{n^2-n+3}$ by the basic comparison test.

There's a better way: The Limit Comparison

Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of positive terms (past some point). Then:

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ then $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ both converge or both diverge.

If $c = 0$, then if $\sum_{n=0}^{\infty} a_n$ diverges, so does $\sum_{n=0}^{\infty} b_n$, and if $\sum_{n=0}^{\infty} b_n$ converges, so does $\sum_{n=0}^{\infty} a_n$.

If $c = \infty$, then if $\sum_{n=0}^{\infty} a_n$ converges, so does $\sum_{n=0}^{\infty} b_n$, and if $\sum_{n=0}^{\infty} b_n$ diverges, so does $\sum_{n=0}^{\infty} a_n$.

ex/ $\sum_{n=0}^{\infty} \frac{1}{n^2-n+3}$.

Look at the dominant term in the numerator (1) and the denominator (n^2).

Compare $\frac{1}{n^2-n+3}$ to $\frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{\frac{1}{n^2-n+3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-n+3} = \lim_{n\to\infty} \frac{1}{1-\cancelto{0}{1/n}+\cancelto{0}{3/n^2}} = \frac{1}{1} = 1.$$

Since $1 > 0$, $\sum_{n=0}^{\infty} \frac{1}{n^2-n+3}$ converges (or diverges) just as $\sum_{n=0}^{\infty} \frac{1}{n^2}$ does.

But, $\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges by the p-test (ie. $p=2>1$) and so:

$$\sum_{n=0}^{\infty} \frac{1}{n^2-n+3} \text{ converges too.}$$

ex/ Does $\sum_{k=2}^{\infty} \frac{1}{\ln(k)}$ converge or diverge?

If it diverges, we need: $\frac{1}{\ln(k)} >$ something that diverges

$\frac{1}{\ln(k)} > \frac{1}{k}$ since $k > \ln(k)$ for $k \geq 2$.

$\sum_{k=2}^{\infty} \frac{1}{k}$ diverges by the p-test (ie. $p = 1 \leq 1$)

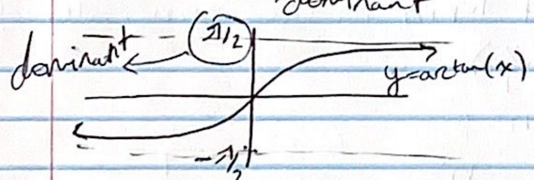
and so $\sum_{k=2}^{\infty} \frac{1}{\ln(k)}$ diverges by the comparison test.

ex/ Does $\sum_{n=0}^{\infty} \frac{\arctan(n)}{1+n^2}$ converge or diverge?

Using the Limit Comparison Test: What dominates?

$\frac{\arctan(n)}{1+n^2} < \frac{\pi/2}{n^2}$ since $1+n^2 > n^2$ and and ...

dominant



But $\sum_{n=0}^{\infty} \frac{\pi/2}{n^2} = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{1}{n^2}$ converges by the p-test (ie. $p = 2 > 1$) and so $\sum_{n=0}^{\infty} \frac{\arctan(n)}{1+n^2}$ converges by the comparison test.

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{\arctan(n)}{n^2+1}} = \frac{n^2+1}{n^2\arctan(n)} = \lim_{n\to\infty} \frac{n^2+1}{n^2\arctan(n)} \cdot \frac{1/n^2}{1/n^2}$$

$$= \lim_{n\to\infty} \frac{1+1/n^2}{\underbrace{\arctan(n)}_{\to \pi/2}} = \frac{1+0}{\pi/2} = 2/\pi > 0$$

So the limit comparison test says $\sum_{n=0}^{\infty} \frac{\arctan(n)}{n^2+1}$ converges if $\sum_{n=0}^{\infty} \frac{1}{n^2}$ does.

Recall: The generalized p-test

$$\sum_{n=0}^{\infty} \frac{a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n + a_0}{b_\ell n^\ell + b_{\ell-1} n^{\ell-1} + \dots + b_1 n + b_0} \quad (\text{where } a_k \neq 0 \text{ or } b_k \neq 0)$$

Converges if $p = \ell - k > 1$ and diverges if $p = \ell - k \leq 1$.

proof/ We'll compare the given series to $\frac{1}{n^p} = \frac{1}{n^{\ell-k}} = \frac{n^k}{n^\ell}$.
Limit Comparison Test:

$$\lim_{n\to\infty} \frac{\left(\frac{a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n + a_0}{b_\ell n^\ell + b_{\ell-1} n^{\ell-1} + \dots + b_1 n + b_0}\right)}{1/n^p}$$

$$= \lim_{n\to\infty} \frac{n^{p \to \ell-k}}{1} \cdot \frac{a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n + a_0}{b_\ell n^\ell + b_{\ell-1} n^{\ell-1} + \dots + b_1 n + b_0}$$

$$= \lim_{n\to\infty} \frac{a_k n^\ell + \dots + a_1 n^{\ell-k+1} + a_0^{\ell-k}}{b_\ell n^\ell + \dots + b_1 n + b_0} \cdot \frac{1/n^\ell}{1/n^\ell}$$

$$= \lim_{n\to\infty} \frac{a_k + \frac{a_{k-1}}{n} + \dots + \frac{a_0^{\ell-k}}{n^\ell}}{b_\ell + \frac{b_{\ell-1}}{n} + \dots + \frac{b_0}{n^\ell}} \quad \begin{matrix} \to 0 \\ \to 0 \end{matrix}$$

$$= \lim_{n\to\infty} \frac{a_k}{b_\ell} \neq 0$$

$$\therefore \sum_{n=0}^{\infty} \frac{a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n + a_0}{b_\ell n^\ell + b_{\ell-1} n^{\ell-1} + \dots + b_1 n + b_0} \text{ converges or}$$

diverges exactly as $\sum_{n=0}^{\infty} \frac{1}{n^p}$ does

ie. as $p = \ell - k > 1$, the series converges
as $p = \ell - k \leq 1$, the series diverges

FIVE STAR.