Series III

More convergence tests ... but first, let's use the integral test a time or two

Recall: (The Integral Test)

- If $f(x)$ is decreasing on $[0,\infty)$ and is integrable then if $a_n = f(n)$ (for $n \geq c$), we have $\sum_{n=c}^{\infty} a_n$ converges exactly when the improper integral $\int_c^{\infty} f(x)\, d(x)$ converges

Q: Dose $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$ converges or not?

$f(x) = \frac{n^2}{2^n} \geq 0$ for $x \geq 0$

$f(x)$ is > 0 for $x > 0$

$$\sum_{n=0}^{\infty} \frac{n^2}{2^n} = \sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

Is $f(x)$ decreasing on $[1,\infty)$?

$$f'(x) = \frac{d}{dx}\left(\frac{x^2}{2^x}\right)$$

$$= \frac{2x(\cancel{2^x}) - x^2(\cancel{2^x} \cdot \ln 2)}{(2^x)^{\cancel{2}}}$$

$$= \frac{2x - x^2(\ln 2)}{2^x}$$

$$= \frac{x(2 - x \cdot \ln 2)}{2^x}$$

So $f'(x) < 0$ exactly when $2 - x \cdot \ln(2) < 0$ exactly when $x \cdot \ln(2) > 2$ ____ , , ____ $x > \frac{2}{\ln(2)}$ $\Downarrow$ $2.885\ldots$

Thus $f(x) < 0$, and so $f(x)$ is decreasing when $x \geq 3$

Thus $\sum_{n=0}^{\infty} \frac{n^2}{2^n} = \sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{1}{2} + \frac{2^2}{2^2} + \sum_{n=3}^{\infty} \frac{n^2}{2^n}$

so the original series converges exactly when $\sum_{n=3}^{\infty} \frac{n^2}{2^n}$ does

But we can apply the integral test to the last series, because $f(x) = \frac{x^2}{2^x}$ satifies the hypothese on $[3,\infty)$

ie $\sum_{n=0}^{\infty}\frac{n^2}{2^n}$ converges as $\sum_{n=3}^{\infty}\frac{n^2}{2^n}$ ____ , ____ which converges as $\int_3^{\infty}\frac{x^2}{2^x}dx$ does:

$$\int_3^{\infty}\frac{x^2}{2^x}dx = \lim_{b\to\infty}\int_3^b \frac{x^2}{2^x}dx$$

$$= \lim_{b\to\infty}\int_3^b x^2\cdot 2^{-x}dx \longrightarrow \quad u = x^2 \quad u' = 2x \quad v' = 2^{-x} \quad v = \frac{-2^{-x}}{\ln(2)}$$

$$= \lim_{b\to\infty}\left[x^2\cdot\frac{-2^{-x}}{\ln(2)}\Big|_3^b - \int_3^b \frac{2x(-2^{-x})}{\ln(2)}dx\right]$$

$$= \lim_{b\to\infty}\left[\frac{-b^2 2^{-b}}{\ln(2)} - \frac{-3^2\cdot 2^{-3}}{\ln(2)} + \frac{2}{\ln(2)}\int_3^b x\cdot 2^{-x}dx\right] \longrightarrow \quad u = x \quad u' = 1 \quad v' = 2^{-x} \quad v = \frac{-2^{-x}}{\ln(2)}$$

$$= \lim_{b\to\infty}\left[\frac{-b^2 2^{-b}}{\ln(2)} + \frac{3^2\cdot 2^{-3}}{\ln(2)} + \frac{2}{\ln(2)}\left(\frac{-2^{-x}\cdot x}{\ln(2)}\Big|_3^b - \int_3^b\frac{-2^{-x}}{\ln(2)}dx\right)\right]$$

$$= \lim_{b\to\infty}\left[\frac{-b^2 2^{-b}}{\ln(2)} + \frac{3^2\cdot 2^{-3}}{\ln(2)} + \frac{2}{\ln(2)}\left(\frac{-b\cdot 2^{-b}}{\ln(2)} - \frac{-3\cdot 2^{-3}}{\ln(2)} + \frac{1}{\ln(2)}\int_3^b 2^{-x}dx\right)\right]$$

$$= \lim_{b\to\infty}\left[\frac{-b^2 2^{-b}}{\ln(2)} + \frac{3^2\cdot 2^{-3}}{\ln(2)} + \frac{-b\cdot 2^{-b+1}}{\ln(2)^2} - \frac{-3\cdot 2^{-3+1}}{\ln(2)^2} + \frac{2}{\ln(2)^2}\cdot\frac{-2^{-x}}{\ln(2)}\Big|_3^b\right]$$

$$= \lim_{b\to\infty}\left[\overset{=0}{\frac{-b^2 2^{-b}}{\ln(2)}} + \cancel{\frac{3^2\cdot 2^{-3}}{\ln(2)}} + \overset{=0}{\frac{-b\cdot 2^{-b+1}}{\ln(2)^2}} - \cancel{\frac{-3\cdot 2^{-3+1}}{\ln(2)^2}} + \overset{=0}{\frac{-2^{-b+1}}{\ln(2)^3}} + \cancel{\frac{-2^{-3+1}}{\ln(2)^3}}\right]$$

- don't need to think about if no "b"

same way as

$$\frac{-b \to \infty}{2^{b-1} \to \infty} \qquad \frac{-1}{2^{b-1}}$$

use L'Hôpital's Rule

$$\lim_{b\to\infty}\frac{b}{2^{b-1}}$$

$$= \lim_{b\to\infty}\frac{1 \to 1}{\ln(2)\cdot 2^{b-1} \to \infty} = 0$$

So the limit exists $= \frac{3^2\cdot 2^{-3}}{\ln(2)} - \frac{-3\cdot 2^{-3+1}}{\ln(2)^2} + \frac{-2^{-3+1}}{\ln(2)^3}$

so $\int_3^b\frac{x^2}{2^{-x}}dx$ coverges and hence so dose $\sum_{n=0}^{\infty}\frac{n^2}{2^n}$