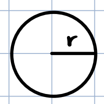
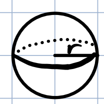
Surface Areas I

of Solids of revolution



$C = 2\pi r$

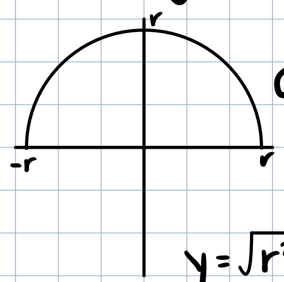
$A = \pi r^2$



$SA = 4\pi r^2$

$V = (4/3)\pi r^3$

How do we get $C = 2\pi r$?



C = 2·arclength of the semi circle

Arc-length of $y = f(x)$, $a \le x \le b$, is

$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = \sqrt{r^2 - x^2}$$

$$y = (r^2 - x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(r^2-x^2)^{-\frac{1}{2}} \cdot \frac{d}{dx}(r^2-x^2)$$

$$= \frac{1}{\cancel{2}}(r^2-x^2)^{-\frac{1}{2}} \cdot (-\cancel{2}x)$$

$$= \frac{-x}{\sqrt{r^2-x^2}}$$

Arc-length of semi-circle of radius r is,

$$\int_{-r}^{r} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_{-r}^{r} \sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\, dx$$

$$= \int_{-r}^{r} \sqrt{1+\frac{x^2}{r^2-x^2}}\, dx$$

$$= \int_{-r}^{r} \sqrt{\frac{r^2-x^2}{r^2-x^2}+\frac{x^2}{r^2-x^2}}\, dx$$

$$= \int_{-r}^{r} \sqrt{\frac{r^2}{r^2-x^2}}\, dx$$

$$= \int_{-r}^{r} r \cdot \frac{1}{\sqrt{r^2-x^2}}\, dx$$

$$= r\int_{-r}^{r} \frac{1}{\sqrt{r^2-x^2}}\, dx \longrightarrow \begin{array}{l} x = r\sin(\theta) \\ dx = r\cos(\theta)\, d\theta \end{array}$$

$$= r\int_{x=-r}^{x=r} \frac{1}{\sqrt{r^2 - r^2\sin^2(\theta)}} \cdot r\cos(\theta)\, d\theta$$

$$= \int_{x=-r}^{x=r} \frac{r^2\cos(\theta)}{\sqrt{r^2\underbrace{(1-\sin^2(\theta))}_{\cos\theta}}}\, d\theta$$

$$= \int_{x=-r}^{x=r} \frac{r^{\cancel{2}}\cancel{\cos(\theta)}}{\cancel{r}\cancel{\cos(\theta)}}\, d\theta$$

$$= \int_{x=-r}^{x=r} r\, d\theta$$

$$= r\theta \Big|_{x=-r}^{x=r}$$

$$= r\cdot\arcsin\left(\frac{x}{r}\right)\Big|_{-r}^{r}$$

$$= r\cdot\arcsin\left(\frac{r}{r}\right) - r\cdot\arcsin\left(\frac{-r}{r}\right)$$

$$= r\cdot\arcsin(1) - r\cdot\arcsin(-1)$$

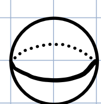
$$= r\left(\frac{\pi}{2}\right) - r\left(\frac{-\pi}{2}\right)$$

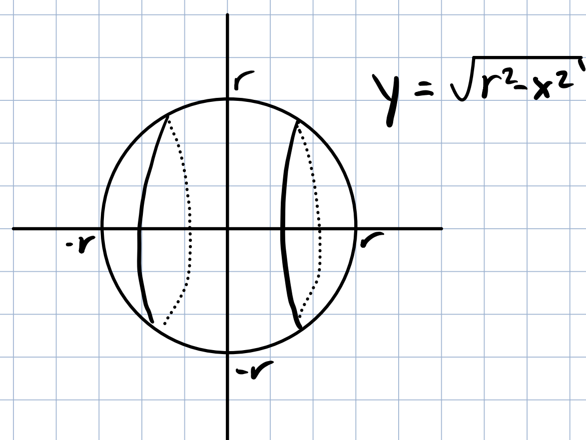
$$= \frac{\pi r}{2} + \frac{\pi r}{2} \quad = \pi r$$

Thus the arclength of a semi-circle of radius r is πr, so the circumference of the entire circle of radius r is $2\pi r$.

Volume of a sphere of radius r,

$$= \frac{4}{3}\pi r^3$$





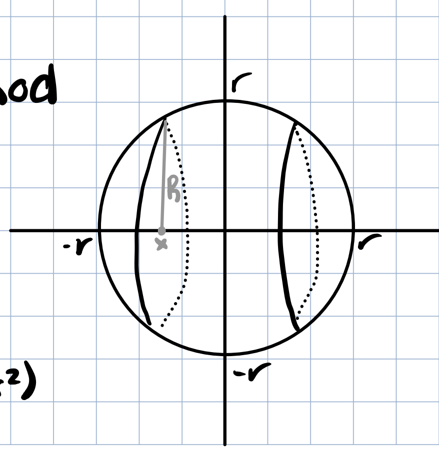
We can get the sphere of radius r by revolving the region between $y=0$ and $y=\sqrt{r^2-x^2}$ (for $-r \le x \le r$) about the x-axis

Using Disk Method

$$R = \sqrt{r^2-x^2} - 0$$

Area of disk at x is

$$\pi R^2 = \pi\left(\sqrt{r^2-x^2}\right)^2 = \pi(r^2-x^2)$$



$$=\pi\int_{-r}^{r}(r^2-x^2)\,dx$$

$$=\pi\left(r^2x-\frac{x^3}{3}\right)\Bigg|_{-r}^{r}$$

$$=\pi\left(r^2\cdot r-\frac{r^3}{3}\right)-\pi\left(r^2(-r)-\frac{(-r)^3}{3}\right)$$

$$=\frac{2}{3}\pi r^3-\left(-\frac{2}{3}\pi r^3\right)$$

$$=\frac{4}{3}\pi r^3$$

Surface area of a sphere should be the sum of the circumference of the disk

$$SA=\int_{-r}^{r}2\pi R\,dx$$

$$=\int_{-r}^{r}2\pi\sqrt{r^2-x^2}\,dx \longrightarrow \begin{array}{l} x=r\sin(\theta) \\ dx=r\cos(\theta) \end{array}$$

x	θ
$-r$	$-\pi/2$
r	$\pi/2$

$$=2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{r^2-r^2\sin^2(\theta)}\cdot r\cos(\theta)\,d\theta$$

$$=2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}r\cos(\theta)\,r\cos(\theta)\,d\theta$$

$$=2\pi r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2(\theta)\,d\theta \longrightarrow \begin{array}{l} \cos 2\theta=2\cos^2(\theta)-1 \\ \cos^2\theta=\frac{1}{2}(1+\cos(2\theta)) \end{array}$$

$$=\cancel{2}\pi r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{\cancel{2}}(1+\cos(2\theta))\,d\theta \longrightarrow \begin{array}{l} u=2\theta \\ \frac{1}{2}du=d\theta \end{array}$$

θ	u
$-\pi/2$	$-\pi$
$\pi/2$	π

$$=\pi r^2\int_{-\pi}^{\pi}(1+\cos(u))\cdot\frac{1}{2}du$$

$$=\frac{\pi r^2}{2}\left(u+\sin(u)\right)\Bigg|_{-\pi}^{\pi}$$

$$=\frac{\pi r^2}{2}\left(\pi+\cancelto{0}{\sin(\pi)}\right)-\frac{\pi r^2}{2}\left(-\pi+\cancelto{0}{\sin(-\pi)}\right)$$

$$= 2\,\frac{\pi^2 r^2}{2}$$

$= \pi^2 r^2 \neq 4\pi r^2$ ... so we're off by a factor of $\frac{4}{\pi}$

What should we have done?