Lecture 9

Feb 8th, 2022

$$\int \frac{f(x)}{g(x)}\,dx = \int \frac{s(x)t(x)+r(x)}{g(x)}\,dx = \int \frac{s(x)t(x)}{g(x)}\,dx + \int \frac{r(x)}{g(x)}\,dx$$

if $f(x)$ degree $\geq g(x)$ degree $\Rightarrow$ Long Divide $g(x)$ into $f(x)$

$\rightarrow$ left with $f(x) = s(x)t(x) + r(x)$
where $r(x)$ degree $\leq g(x)$.

$$\int \frac{ax+b}{(cx+d)^n}\,dx \quad \text{or} \quad \int \frac{ax+b}{cx^2+dx+e}\,dx$$

can be solved by combining algebra and substitutions.

ex/ $\int \frac{x-1}{3x^2+2x+4}\,dx$

$u = 3x^2+2x+4$
$\Rightarrow du = (6x+2)\,dx$
$\Rightarrow \frac{1}{6}du = (x+\frac{1}{3})\,dx$

$$= \int \frac{x+\frac{1}{3}-\frac{1}{3}-1}{3x^2+2x+4}\,dx$$

$$= \int \frac{x+\frac{1}{3}}{3x^2+2x+4}\,dx - \int \frac{4/3}{3x^2+2x+4}\,dx$$

$$= \int \frac{1}{u}\cdot\frac{1}{6}\,du - \frac{4}{3}\int \frac{1}{3x^2+2x+4}\,dx$$

$3x^2+2x+4$
$= 3(x^2+\frac{2}{3}x+\frac{4}{3})$
$= 3((x+\frac{1}{3})^2 - \frac{1}{9} + \frac{4}{3})$
$= 3((x+\frac{1}{3})^2 + \frac{11}{9})$

$$= \frac{1}{6}\ln(u) - \frac{4}{3}\int \frac{1}{3[(x+\frac{1}{3})^2+\frac{11}{9}]}\,dx$$

$$= \frac{1}{6}\ln(u) - \frac{4}{9}\int \frac{1}{(x+\frac{1}{3})^2+\frac{11}{9}}\,dx$$

$w = x+\frac{1}{3}$
$dw = dx$

$$= \frac{1}{6}\ln(3x^2+2x+4) - \frac{4}{9}\int \frac{1}{w^2+\frac{11}{9}}\,dw$$

$w = \frac{\sqrt{11}}{3}\tan(\theta)$
$dw = \sqrt{\frac{11}{9}}\sec^2(\theta)\,d\theta$

$$= \frac{1}{6}\ln(3x^2+2x+4) - \frac{4}{9}\int \frac{1}{(\sqrt{\frac{11}{9}}\tan(\theta))^2+\frac{11}{9}}\cdot\sqrt{\frac{11}{9}}\sec^2(\theta)\,d\theta$$

$$= \frac{1}{6}\ln(3x^2+2x+4) - \frac{4}{9}\int \frac{1}{\frac{11}{9}(\tan^2(\theta)+1)}\cdot\sqrt{\frac{11}{9}}\sec^2(\theta)\,d\theta$$

$$= \frac{1}{6}\ln(3x^2+2x+4) - \frac{4}{9}\cdot\frac{1}{\sqrt{11/9}}\int \frac{\sec^2(\theta)}{\sec^2(\theta)}\,d\theta$$

$$= \frac{1}{6}\ln(3x^2+2x+4) - \frac{4}{3\sqrt{11}}\theta + C = \frac{1}{6}\ln(3x^2+2x+4) - \frac{4}{3\sqrt{11}}\arctan\left(\frac{w}{\sqrt{11/9}}\right) + C$$

$$= \frac{1}{6}\ln(3x^2+2x+4) - \frac{4}{3\sqrt{11}}\arctan\left(\frac{x+1/3}{\sqrt{11/9}}\right) + C = \frac{1}{6}\ln(3x^2+2x+4) - \frac{4}{3\sqrt{11}}\arctan\left(\frac{3x+1}{\sqrt{11}}\right) + C$$

ex/ $\displaystyle\int \frac{x^2+3x+41}{(x^2+1)(x+1)}\,dx$

$$\frac{Ax+b}{x^2+1} + \frac{C}{x+1}$$

$$= \frac{(Ax+b)(x+1) + C(x^2+1)}{(x^2+1)(x+1)}$$

$$= \frac{Ax^2+Ax+bx+b+Cx^2+C}{(x^2+1)(x+1)}$$

$$= \frac{(A+C)x^2+(A+b)x+(b+C)}{(x^2+1)(x+1)}$$

$$\Rightarrow \begin{array}{l} A+C=1 \\ A+b=3 \\ b+C=41 \end{array} \quad \begin{array}{l} \Rightarrow A = 1-C \\ \Rightarrow 1-C+b=3 \Rightarrow b = 2+C \\ \Rightarrow 2+C+C=41 \end{array}$$

$$\Rightarrow C = \frac{39}{2}$$

$$\Rightarrow b = 2 + \frac{39}{2} = \frac{43}{2}$$

$$\Rightarrow A = 1 - \frac{39}{2} = \frac{-37}{2}$$

$$= \int \frac{-\frac{37}{2}x + \frac{43}{2}}{x^2+1}\,dx + \int \frac{39/2}{x+1}\,dx$$

$\downarrow$

Now Solve using Previous methods.

$$\int \frac{x^2+x+1}{\underbrace{(ax^2+bx+c)^3}_{\text{assume irreducible}}\underbrace{(dx^2+ex+f)}(x-g)^2}\,dx$$

$$= \int \frac{Ax+B}{(ax^2+bx+c)^3}\,dx + \int \frac{Cx+D}{(ax^2+bx+c)^2}\,dx + \int \frac{Ex+F}{ax^2+bx+c}\,dx$$

$$+ \int \frac{Gx+H}{dx^2+ex+f}\,dx + \int \frac{I}{(x-g)^2}\,dx + \int \frac{J}{x-g}\,dx$$

* use mathematical software to solve for variables A – J

* break it up according to a complete factorization of the denominator and step down from biggest power of each factor.

• we can factor quadratics using QRF

$$3x^2+2x+4=0 \Rightarrow x = \frac{-2 \pm \sqrt{2^2-4(3)(4)}}{2(3)} \Rightarrow x = \frac{-2\sqrt{-44}}{6}$$

irreducible

$$3x^2+2x-4=0 \Rightarrow x = \frac{-2 \pm \sqrt{2^2-4(3)(-4)}}{2(3)} \Rightarrow x = \frac{-2 \pm \sqrt{52}}{6}$$

$$\hookrightarrow \text{so } 3x^2+2x-4 = \left(x - \frac{-2+\sqrt{52}}{6}\right)\left(x + \frac{-2-\sqrt{52}}{6}\right) \cdot 3$$

• There are no general factorization formulae after the 4th degree.

ex/ $$\int \frac{1}{x^4-1}\,dx = \int \frac{1}{(x^2+1)(x^2-1)}\,dx = \int \frac{1}{(x^2+1)(x+1)(x-1)}\,dx$$

$$\Rightarrow \frac{1}{(x^2+1)(x+1)(x-1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$= \frac{A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)}{(x-1)(x+1)(x^2+1)}$$

$$= \frac{A(x^3+x^2+x+1) + B(x^3-x^2+x-1) + (Cx+D)(x^2-1)}{(x-1)(x+1)(x^2+1)}$$

$$= \frac{Ax^3+Ax^2+Ax+A+Bx^3-Bx^2+Bx-B+Cx^3+Dx^2-Cx-D}{(x-1)(x+1)(x^2+1)}$$

$$= \frac{(A+B+C)x^3+(A-B+D)x^2+(A+B-C)x+(A-B-D)}{(x-1)(x+1)(x^2+1)}$$

$\Rightarrow$ ① $A+B+C=0$ ② $A-B+D=0$ ③ $A+B-C=0$ ④ $A-B-D=1$

② − ④ $\Rightarrow 2D=-1 \Rightarrow D=-\frac{1}{2}$ *

① − ③ $\Rightarrow 2C=0 \Rightarrow \underline{C=0}$ *

① $A+B=0$ ② $A-B=\frac{1}{2}$

$A=-B$ $\nearrow$ $-B-B=\frac{1}{2} \Rightarrow -2B=\frac{1}{2} \Rightarrow \underline{B=-\frac{1}{4}}$ *

$\Rightarrow \underline{A=\frac{1}{4}}$ *

$$\Rightarrow \int \frac{1}{x^4-1}\,dx = \int \frac{A}{x-1}\,dx + \int \frac{B}{x+1}\,dx + \int \frac{Cx+D}{x^2+1}\,dx$$

$$= \int \frac{1/4}{x-1}\,dx + \int \frac{-1/4}{x+1}\,dx + \int \frac{-1/2}{x^2+1}\,dx$$

$$= \frac{1}{4}\int \frac{1}{x-1}\,dx - \frac{1}{4}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$u = x-1$, $du = dx$
$w = x+1$, $dw = dx$

$$= \frac{1}{4}\int \frac{1}{u}\,du - \frac{1}{4}\int \frac{1}{w}\,dw - \frac{1}{2}\arctan(x)$$

$$= \frac{1}{4}\ln(u) - \frac{1}{4}\ln(w) - \frac{1}{2}\arctan(x) + C$$

$$= \frac{1}{4}\ln(x-1) - \frac{1}{4}\ln(x+1) - \frac{1}{2}\arctan(x) + C$$