Trogonometric Substitutions III

How to handle $\int \frac{1}{\sqrt{\text{random quadratic}}}\,dx$

ex: $\int \frac{1}{\sqrt{4x^2+2x+1}}\,dx \longrightarrow 4x^2+2x+1 \Rightarrow 4(x^2+\frac{1}{2}x+\frac{1}{4})$

$= \int \frac{1}{2\sqrt{x^2+\frac{1}{2}x+\frac{1}{4}}}\,dx$

$= \frac{1}{2}\int \frac{1}{\sqrt{x^2+\frac{1}{2}x+\frac{1}{4}}}\,dx \longrightarrow x^2+\frac{1}{2}x+\frac{1}{4} \Rightarrow$ complete the square (x^2+bx+c)

$\Rightarrow (x+\frac{b}{2})^2-(\frac{b}{2})^2+c$

$\Rightarrow x^2+2\frac{b}{2}x+\frac{b^2}{4}-\frac{b^2}{4}+c$

$\longrightarrow (x+\frac{1}{2}\cdot\frac{1}{2})^2-\left(\frac{\frac{1}{2}}{2}\right)^2+\frac{1}{4}$

$\longrightarrow (x+\frac{1}{4})^2+\frac{1}{8}$

$= \frac{1}{2}\int \frac{1}{\sqrt{(x+\frac{1}{4})^2+\frac{1}{8}}}\,dx \longrightarrow u = x+\frac{1}{4},\ du = dx$

$= \frac{1}{2}\int \frac{1}{\sqrt{u^2+\frac{1}{8}}} \longrightarrow u = \frac{1}{\sqrt{8}}\tan\theta,\ du = \frac{1}{\sqrt{8}}\sec^2\theta\,d\theta$

$= \frac{1}{2}\int \frac{1}{\sqrt{(\frac{1}{\sqrt{8}}\tan\theta)^2+\frac{1}{8}}}\cdot\frac{1}{\sqrt{8}}\sec^2\theta\,d\theta$

$= \frac{1}{2}\cdot\frac{1}{\sqrt{8}}\int \frac{\sec^2\theta}{\sqrt{\frac{1}{8}\tan^2\theta+\frac{1}{8}}}\,d\theta$

$= \frac{1}{2\sqrt{8}}\int \frac{\sec^2\theta}{\sqrt{\frac{1}{8}}\sqrt{\tan^2\theta+1}}\,d\theta$

$= \frac{1}{2\cancel{\sqrt{8}}\cancel{\sqrt{\frac{1}{8}}}}\int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\,d\theta$

$= \frac{1}{2}\int \sec\theta\,d\theta$

$= \frac{1}{2}\ln(\sec\theta+\tan\theta)+C \longrightarrow$ undo our subsitutions, $u = \frac{1}{\sqrt{8}}\tan\theta$ and $u = x+\frac{1}{4}$

$\Rightarrow \tan\theta = \sqrt{8}\,u$

$\sec\theta = \sqrt{1+\tan^2\theta}$

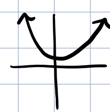
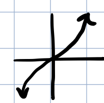
$= \sqrt{1+8u^2}$

$= \frac{1}{2}\ln(\sqrt{1+8u^2}+\sqrt{8}\,u)+C$

$= \frac{1}{2}\ln(\sqrt{1+8(x+\frac{1}{4})^2}+\sqrt{8}(x+\frac{1}{4}))+C$

An alternative to trig functions

The hyperbolic functions reminder

$\cosh(x) = \dfrac{e^x+e^{-x}}{2}$  $\sinh(x) = \dfrac{e^x-e^{-x}}{2}$ 

$$\cosh^2(x) = \frac{e^{2x}+2e^xe^{-x}+e^{-2x}}{4} = \frac{e^{2x}+2+e^{-2x}}{4}$$

$$\sinh^2(x) = \frac{e^{2x}-2e^xe^{-x}+e^{-2x}}{4} = \frac{e^{2x}-2+e^{-2x}}{4}$$

$$\cosh^2(x)-\sinh^2(x) = \frac{2-(-2)}{4} = 1$$

$\dfrac{d}{dx}\cosh(x) = \sinh(x)$ $\dfrac{d}{dx}\sinh(x) = \cosh(x)$

If you see $\sqrt{1+x^2}$, you can try $x = \sinh(t)$

$\sqrt{x^2-1}$, you can try $x=\cosh(t)$