Integrating (mixed powers of) trigonometric functions II

When faced with something like

$$\int \sin^3(x)\cos^2(x)\,dx \quad \text{or} \quad \int \sec^3(x)\tan^3(x)\,dx,$$

usually the best strategy is to change all or most of one function to the other using trig identities, to set things up for a substitution or a reduction formula.

Eg $\int \sin^3(x)\cos^2(x)\,dx \overset{①}{=} \int \sin(x)\sin^2(x)\cos^2(x)\,dx$ $\left[\begin{array}{l}\cos^2(x)+\sin^2(x)=1 \\ \Rightarrow \sin^2(x)=1-\cos^2(x)\end{array}\right]$

$= \int \sin(x)\left(1-\cos^2(x)\right)\cos^2(x)\,dx$ $\quad u=\cos(x)$, $du=-\sin(x)\,dx$, $(-1)\,du=\sin(x)\,dx$

$= \int (1-u^2)u^2(-1)\,du$

$= \int (u^4-u^2)\,du = \frac{u^5}{5} - \frac{u^3}{3} + C$

$= \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C$

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$$\int \sin^3(x)\cos^2(x)\,dx \overset{②}{=} \int \sin^3(x)\left(1-\sin^2(x)\right)dx \qquad \left[1-\sin^2(x) = \cos^2(x)\right]$$

$$= \int \left[\sin^3(x) - \sin^5(x)\right] dx$$

$$= \int \sin^3(x)\,dx - \int \sin^5(x)\,dx$$

Use the reduction formula for $\int \sin^n(x)\,dx$:

$$= -\frac{1}{3}\sin^{3-1}(x)\cos(x) + \frac{3-1}{3}\int \sin^{3-2}(x)\,dx$$

$$- \left[-\frac{1}{5}\sin^{5-1}(x)\cos(x) + \frac{5-1}{5}\int \sin^{5-2}(x)\,dx\right]$$

$$= \left[-\frac{1}{3}\sin^2(x)\cos(x) + \frac{2}{3}\left(-\cos(x)\right)\right] - \left[-\frac{1}{5}\sin^4(x)\cos(x) + \frac{4}{5}\int \sin^3(x)\,dx\right]$$

$$= \left[-\frac{1}{3}\sin^2(x)\cos(x) - \frac{2}{3}\cos(x)\right] - \left[-\frac{1}{5}\sin^4(x)\cos(x) + \frac{4}{5}\left(-\frac{1}{3}\sin^2(x)\cos(x) - \frac{2}{3}\cos(x)\right)\right] + C$$

$$= -\underset{=\frac{5}{15}}{\frac{1}{3}}\sin^2(x)\cos(x) - \underset{=\frac{10}{15}}{\frac{2}{3}}\cos(x) + \frac{1}{5}\sin^4(x)\cos(x) + \frac{4}{15}\sin^2(x)\cos(x) + \frac{8}{15}\cos(x) + C$$

$$= \frac{1}{5}\sin^4(x)\cos(x) - \frac{1}{15}\sin^2(x)\cos(x) - \frac{2}{15}\cos(x) + C.$$

Note that method ① requires less work, but you need to spot a little more [ie the substitution].

What about our other example?

$$\int \sec^3(x)\tan^3(x)\,dx$$

You can't just convert all the sec(x) (or tan(x)) into the other function: the powers are odd and the key identity, $1+\tan^2(x)=\sec^2(x)$, has even powers.

$$= \int \sec^2(x)\tan^2(x)\left(\sec(x)\tan(x)\right)dx$$

& note that $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$

Set this up for the substitution $w=\sec(x)$, so $dw=\sec(x)\tan(x)dx$

$$= \int \sec^2(x)\left(-1+\sec^2(x)\right)\sec(x)\tan(x)\,dx = \int w^2(w^2-1)\,dw$$

$$= \int (w^4-w^2)\,dw = \frac{w^5}{5} - \frac{w^3}{3} + C = \frac{1}{5}\sec^5(x) - \frac{1}{3}\sec^3(x) + C$$

What if we mix sin(x) or cos(x) with sec(x) or tan(x) (or cot(x) & csc(x))? What most often works is putting everything in terms of sin(x) & cos(x) and then looking for ways to simplify & rearrange using algebra & trig identities until we get something we can handle.

es $\displaystyle\int \sin(x)\tan^3(x)\,dx = \int \frac{\sin^4(x)}{\cos^3(x)}\,dx$ since $\sin(x)\left(\frac{\sin(x)}{\cos(x)}\right)^3 = \sin(x)\tan^3(x)$

We have an even power of sin(x) so we rewrite this in terms of cos(x)

$$= \int \frac{\left(\sin^2(x)\right)^2}{\cos^3(x)}\,dx = \int \frac{\left(1-\cos^2(x)\right)^2}{\cos^3(x)}\,dx$$

$$= \int \frac{1-2\cos^2(x)+\cos^4(x)}{\cos^3(x)}\,dx = \int \left(\frac{1}{\cos^3(x)} - \frac{2}{\cos(x)} + \cos(x)\right)dx$$

$$= \int \left(\sec^3(x) - 2\sec(x) + \cos(x)\right)dx$$

$$= \int \sec^3(x)\,dx - 2\int \sec(x)\,dx + \int \cos(x)\,dx$$

$$= \left[\frac{1}{3-1}\sec^{3-2}(x)\tan(x) + \frac{3-2}{3-1}\int \sec^{3-2}(x)\,dx\right] - 2\int \sec(x)\,dx + \int \cos(x)\,dx$$

$$= \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\,dx - 2\int \sec(x)\,dx + \int \cos(x)$$

$$= \frac{1}{2}\sec(x)\tan(x) - \frac{3}{2}\int \sec(x)\,dx + \int \cos(x)$$

$$= \frac{1}{2}\sec(x)\tan(x) - \frac{3}{2}\ln\left(\sec(x)+\tan(x)\right) + \sin(x) + C$$

We ought be awake to the possibilities for simplification...

$$\int \sin(x)\tan^3(x)\,dx = \int \frac{\sin^4(x)}{\cos^3(x)}\,dx = \int \frac{\sin^4(x)}{(1-\sin^2(x))\cos(x)}\,dx$$

& now what?

... but not all ideas that might work actually do.

eg Let's look at one example where there are really good opportunities to simplify:

$$\int \frac{\sec^2(x)-\tan^2(x)}{\cos^4(x)-\sin^4(x)}\,dx = \int \frac{1}{(\cos^2(x)-\sin^2(x))\underbrace{(\cos^2(x)+\sin^2(x))}_{=1}}\,dx$$

$1+\tan^2(x) = \sec^2(x)$

$\Rightarrow 1 = \sec^2(x)-\tan^2(x)$

& $\cos^2(x)+\sin^2(x) = 1$

$$= \int \frac{1}{\cos^2(x)-\sin^2(x)}\,dx$$

$$= \int \frac{1}{(\cos(x)-\sin(x))(\cos(x)+\sin(x))}\,dx$$

Not promising.

But

$\cos^2(x) - \sin^2(x) = \cos(2x)$, so
$$= \int \frac{1}{\cos(2x)}\, dx$$

$$= \int \sec(2x)\, dx \qquad \begin{array}{l} w = 2x \\ dw = 2dx \end{array} \Rightarrow dx = \tfrac{1}{2}\, dw$$

$$= \int \sec(w)\, \tfrac{1}{2}\, dw = \tfrac{1}{2} \int \sec(w)\, dw$$

$$= \tfrac{1}{2} \ln\left(\sec(w) + \tan(w)\right) + C$$

$$\boxed{= \tfrac{1}{2} \ln\left(\sec(2x) + \tan(2x)\right) + C}$$

$$= \tfrac{1}{2} \ln\left(\sqrt{\sec(2x) + \tan(2x)}\right) + C.$$

Next time: trigonometric substitutions
(& hyperbolic substitutions)