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Integrating (powers of) trigonometric functions

Last time we used integration by parts to obtain a "reduction formula" for integrating powers of $\cos(x)$.

$$(n \geq 2) \qquad \int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx.$$

Similarly, we have reduction formulas for $\sin(x)$, $\sec(x)$ & $\tan(x)$. (and $\cot(x)$ & $\csc(x)$, but these are rarely used). Need $n \geq 2$ for all of these:

$$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

$$\int \sec^n(x)\,dx = \frac{1}{n-1}\tan(x)\sec^{n-2}(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

$$\int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx$$

We'll derive the last one and then do some examples, also doing them in alternate ways.

②

$(n \geq 2) \int \tan^n(x)\,dx$

Use $1 + \tan^2(x) = \sec^2(x)$,
so $\tan^2(x) = \sec^2(x) - 1$

$= \int \tan^{n-2}(x) \cdot \tan^2(x)\,dx = \int \tan^{n-2}(x)\,(\sec^2(x) - 1)\,dx$

$= \int \left(\tan^{n-2}(x)\sec^2(x) - \tan^{n-2}(x)\right)dx$

$= \int \tan^{n-2}(x)\sec^2(x)\,dx - \int \tan^{n-2}(x)\,dx$

$w = \tan(x)$
$dw = \sec^2(x)\,dx$

$= \int w^{n-2}\,dw - \int \tan^{n-2}(x)\,dx$

$= \dfrac{w^{n-1}}{n-1} - \int \tan^{n-2}(x)\,dx$

$= \dfrac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx$ ✓

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$$\int_{-\pi/2}^{\pi/2} \cos^3(x)\,dx = \frac{1}{3}\cos^{3-1}(x)\sin(x)\Big|_{-\pi/2}^{\pi/2} + \frac{3-1}{3}\int_{-\pi/2}^{\pi/2}\cos^{3-2}(x)\,dx$$

Using the reduction formula:

$$= \frac{1}{3}\cos^2(x)\sin(x)\Big|_{-\pi/2}^{\pi/2} + \frac{2}{3}\int_{-\pi/2}^{\pi/2}\cos(x)\,dx$$

$$= \frac{1}{3}\underbrace{\cos^2\left(\tfrac{\pi}{2}\right)}_{0^2}\cdot\underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{1} - \frac{1}{3}\underbrace{\cos^2\left(\tfrac{-\pi}{2}\right)}_{0^2}\cdot\underbrace{\sin\left(-\tfrac{\pi}{2}\right)}_{-1} + \frac{2}{3}\sin(x)\Big|_{-\pi/2}^{\pi/2}$$

$$= 0 - 0 + \frac{2}{3}\overbrace{\sin\left(\tfrac{\pi}{2}\right)}^{1} - \frac{2}{3}\overbrace{\sin\left(-\tfrac{\pi}{2}\right)}^{-1}$$

$$= \frac{2}{3} - \left(\frac{2}{3}\right)(-1) = \frac{4}{3}$$

Using the identity

$\sin^2(x) + \cos^2(x) = 1$

$\Rightarrow \cos^2(x) = 1 - \sin^2(x)$

& the substitution

$w = \sin(x)$

$dw = \cos(x)\,dx$

x	w
$-\frac{\pi}{2}$	-1
$\frac{\pi}{2}$	1

$$= \int_{-\pi/2}^{\pi/2}\cos^2(x)\cos(x)\,dx = \int_{-\pi/2}^{\pi/2}(1-\sin^2(x))\cos(x)\,dx$$

$$= \int_{-1}^{1}(1-w^2)\,dw = \left(w - \frac{w^3}{3}\right)\Big|_{-1}^{1}$$

$$= \left(1 - \frac{1^3}{3}\right) - \left(-1 - \frac{(-1)^3}{3}\right)$$

$$= \left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right) = 1 + 1 - \frac{1}{3} - \frac{1}{3} = \frac{6}{3} - \frac{2}{3} = \frac{4}{3}$$ ✓

$$\int \sec^4(x)\,dx = \frac{1}{4-1}\tan(x)\sec^{4-2}(x) + \frac{4-2}{4-1}\int \sec^{4-2}(x)\,dx$$

Using the reduction formula.

$$= \frac{1}{3}\tan(x)\sec^2(x) + \frac{2}{3}\int \sec^2(x)\,dx$$

$$= \frac{1}{3}\tan(x)\sec^2(x) + \frac{2}{3}\tan(x) + C$$

Using $\sec^2(x) = 1+\tan^2(x)$ & the substitution $w = \tan(x)$, so $dw = \sec^2(x)\,dx$

$$= \int \sec^2(x)\sec^2(x)\,dx = \int (1+\tan^2(x))\sec^2(x)\,dx$$

$$= \int (1+w^2)\,dw = \left(w + \frac{w^3}{3}\right) + C$$

$$= \tan(x) + \frac{1}{3}\tan^3(x) + C$$

This is the same as the previous one:

$$\tan(x) + \frac{1}{3}\tan^3(x) + C = \tan(x) + \frac{1}{3}\tan(x)\tan^2(x) + C$$

$$= \tan(x) + \frac{1}{3}\tan(x)(\sec^2(x) - 1) + C$$

$$= \tan(x) + \frac{1}{3}\tan(x)\sec^2(x) - \frac{1}{3}\tan(x) + C$$

$$= \frac{1}{3}\tan(x)\sec^2(x) - \frac{2}{3}\tan(x) + C$$ ✓

$$\int \sin^5(x)\,dx = -\frac{1}{5}\sin^{5-1}(x)\cos(x) + \frac{5-1}{5}\int \sin^{5-2}(x)\,dx$$

$$= -\frac{1}{5}\sin^4(x)\cos(x) + \frac{4}{5}\int \sin^3(x)\,dx$$

Using the reduction formula.

$$= -\frac{1}{5}\sin^4(x)\cos(x) + \frac{4}{5}\left[-\frac{1}{3}\sin^{3-1}(x)\cos(x) + \frac{3-1}{3}\int \sin^{3-2}(x)\,dx\right]$$

$$= -\frac{1}{5}\sin^4(x)\cos(x) + \frac{4}{5}\left[-\frac{1}{3}\sin^2(x)\cos(x) + \frac{2}{3}\int \sin(x)\,dx\right]$$

$$= -\frac{1}{5}\sin^4(x)\cos(x) - \frac{4}{15}\sin^2(x)\cos(x) + \frac{8}{15}(-\cos(x)) + C$$

$$= -\frac{1}{5}\sin^4(x)\cos(x) - \frac{4}{15}\sin^2(x)\cos(x) - \frac{8}{5}\cos(x) + C$$

$$= -\frac{1}{5}\cos(x)\left[\sin^4(x) + \frac{4}{3}\sin^2(x) + 8\right] + C$$

Use $\sin^2(x) = 1 - \cos^2(x)$ & the substitution

$w = \cos(x)$

$dw = -\sin(x)\,dx$

$\Rightarrow (-1)\,dw = \sin(x)\,dx$

$$= \int \sin^4(x)\sin(x)\,dx = \int (\sin^2(x))^2 \sin(x)\,dx$$

$$= \int (1-\cos^2(x))^2 \sin(x)\,dx = (-1)\int (1-w^2)^2\,dw$$

$$= (-1)\int (1-2w^2+w^4)\,dw = (-1)\left(w - \frac{2}{3}w^3 + \frac{w^5}{5}\right) + C$$

$$= -\cos(x) + \frac{2}{3}\cos^3(x) - \frac{1}{5}\cos^5(x) + C.$$

Exercise: Show these answers are the same.

You can integrate $\int \cos^2(x)\,dx$ or $\int \sin^2(x)\,dx$ in one other way: (ie besides using the reduction formula)

Using the identities

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$
$$= 2\cos^2(x) - 1$$
$$= 1 - 2\sin^2(x)$$

$$\Rightarrow \quad \cos^2(x) = \frac{1}{2}\left(1 + \cos(2x)\right)$$

$$\& \quad \sin^2(x) = \frac{1}{2}\left(1 - \cos(2x)\right)$$

Thus $\int \cos^2(x)\,dx = \frac{1}{2}\int (1 + \cos(2x))\,dx$ & now substitute $u = 2x$, so $dx = \frac{1}{2}du$.

& $\int \sin^2(x)\,dx = \frac{1}{2}\int (1 - \cos(2x))\,dx$ — " —

From time to time we could also use the identity $\sin(2x) = 2\sin(x)\cos(x)$.