The Fundamental Theorem of Calculus

or, using antiderivatives to compute definite integrals.

We saw that using the Left-Hand Rule to actually compute a definite integral is pretty painful even for $\int_0^2 (1-x^2)\,dx$.

Fortunately there is usually a better way, namely using antiderivatives,

eg $$\int_0^2 (1-x^2)\,dx = \underbrace{\left(x - \frac{x^3}{3}\right)}_{\text{antiderivative of } 1-x^2}\Bigg|_0^2$$

$$= \left(2 - \frac{2^3}{3}\right) - \cancel{\left(0 - \frac{0^3}{3}\right)}$$

$$= 2 - \frac{8}{3} = -\frac{2}{3}$$

Note that defining the definite integral in this way does not suffice to get basic properties integration like $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ work properly. See Assignment #1!

We have two versions of the Fundamental Theorem:

I. Suppose that $F(x)$ is a function on $[a,b]$ which is differentiable at every point x in $[a,b]$ and such that $f(x) = F'(x)$ is bounded [ie no vertical asymptote] ~~for x in~~ ~~[a,b]~~ on (a,b).

Then $\int_a^b f(x)\,dx = F(b) - F(a)$,

We'll mostly use this one.

assuming that $f(x)$ is <u>integrable</u> on $[a,b]$, <u>ie</u> $\int_a^b f(x)\,dx$ makes sense.

II. Suppose that $f(x)$ is defined and integrable (and bounded) on $[a,b]$. Then if $F(x) = \int_a^x f(t)\,dt$ $(x \in [a,b])$,

then $F(x)$ is differentiable on (a,b) and $F'(x) = f(x)$.

This one is important for defining ~~certain~~ certain functions & solving differential equations.

To use this in practice, we need to develop

- a library of basic antiderivatives

  [run various basic derivatives in reverse]

&

- a suite of techniques & properties that

  let us reduce various integrals to the basic ones.

We have a basic start on such a library by considering what we know of derivatives:

eg $\int \sin(x)\,dx = -\cos(x) + C$

since $\frac{d}{dx}\left(-\cos(x) + C\right) = -(-\sin(x)) + 0 = \sin(x)$

Note that if $F(x) = G(x) + C$ where C is a constant, then $F'(x) = G'(x)$, so both are antiderivatives for $f(x)$. So when we deal with solving indefinite integrals $\int f(x)\,dx$ (no limits!), ie finding antiderivatives we ought to put in a generic constant like C, ...

List:

1. $\int \sin(x)\,dx = -\cos(x) + C$

2. $\int \cos(x)\,dx = \sin(x) + C$

3. $\int \sec^2(x)\,dx = \tan(x) + C$

4. $\int e^x\,dx = e^x + C$

$(a>0)$ 5. $\int a^x\,dx = \dfrac{a^x}{\ln(a)} + C$ since $\frac{d}{dx} a^x = \ln(a)\cdot a^x$

Power Rule

6. $$\int x^n\,dx = \begin{cases} \dfrac{x^{n+1}}{n+1} & \text{if } n \neq -1 \\ \ln(n) & \text{if } n = -1 \end{cases}$$

since $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \cancel{\frac{1}{n+1}} \cdot \cancel{(n+1)}\, x^{n+\cancel{1-1}} = x^n$ $\left(\begin{matrix}\text{if} \\ n\neq -1\end{matrix}\right)$

& $\frac{d}{dx} \ln(x) = \frac{1}{x} = x^{-1}$

7. $\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$ since $\frac{d}{dx}\arctan(x)$

$= \tan^{-1}(x) + C$ $= \frac{1}{1+x^2}$

Techniques & Properties

1. Power Rule

2. $\int c f(x)\,dx = c\int f(x)\,dx$ if c is a constant

3. $\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

   (& similarly for $-$ instead of $+$)

4. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

5. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

Since 0 is the only number for which $-0 = 0$, we have $\int_a^a f(x)\,dx = -\int_a^a f(x)\,dx$
$= 0$.

We'll build on our library of techniques & use that to expand our list of basic antiderivatives.