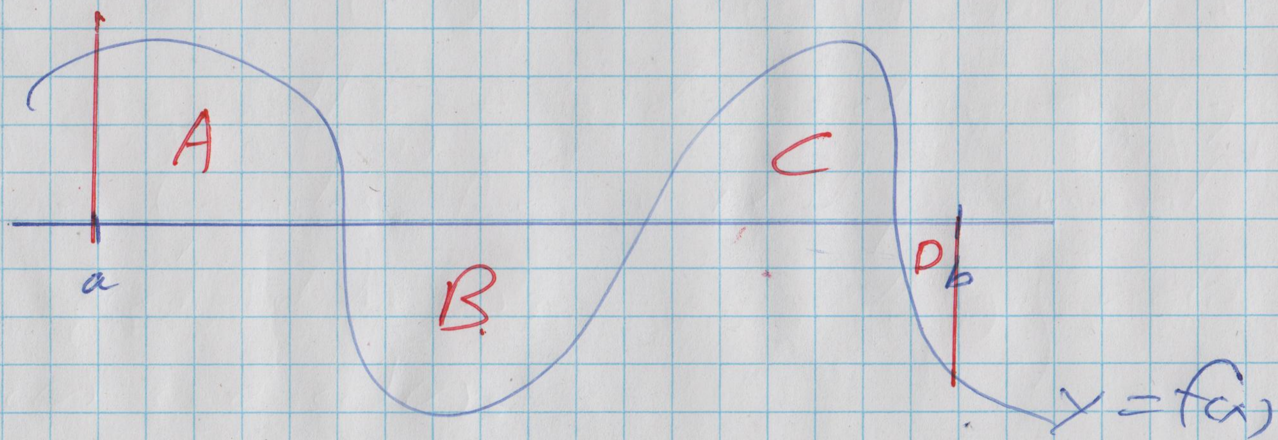
The Definite Integral,

or, computing areas

Short version: The definite integral $\int_a^b f(x)\,dx$ ("the integral of f(x) from a to b")

represents the weighted area between $y = f(x)$ and the x-axis between $x = a$ and $x = b$.

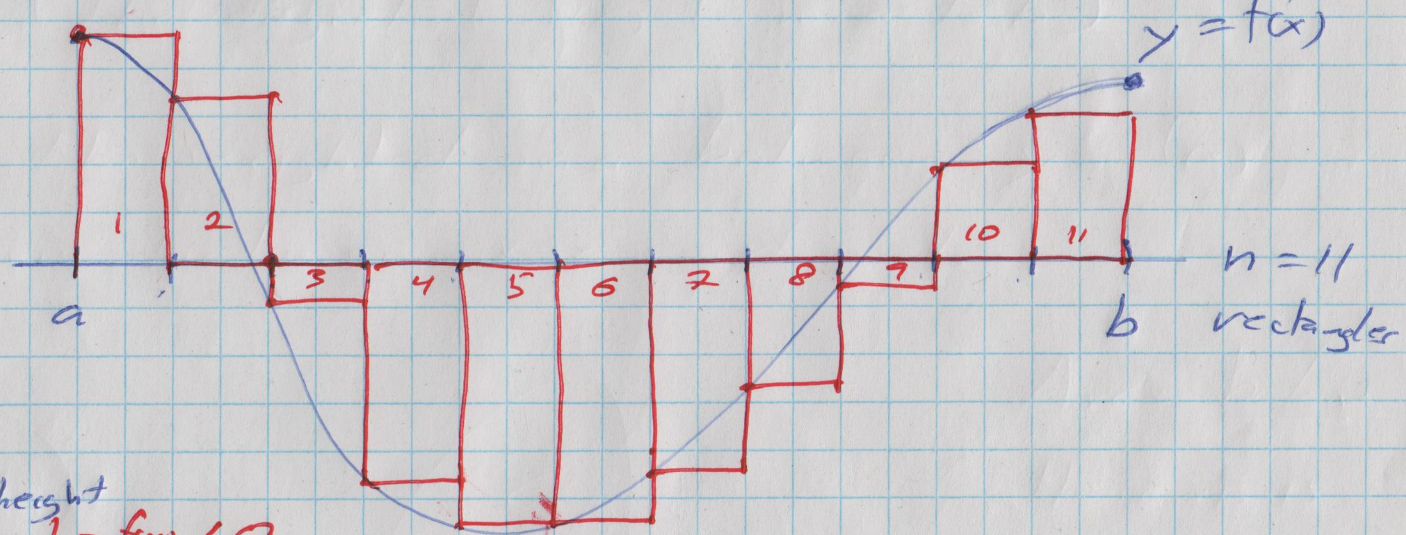


Area below the axis is negative area and area above is positive area

$$\int_a^b f(x)\,dx = A - B + C - D$$

How do we actually define this a little more precisely? Our textbook uses the Left-Hand Rule (even if it doesn't call it that):

1° Divide $[a,b]$ into equal sub-intervals



Note: if $f(x)$ is negative at a left-hand endpoint, the area of the rectangle ~~bh~~ base · height will be negative since $h = f(x) < 0$.

2° Aproximate the area by building a rectangle on each sub-interval that has height given by $f(x)$ at the left-hand endpoint of each subinterval.

3° As $n \to \infty$ the areas of the rectangles approximate $\int_a^b f(x)dx$ better & better.

Note that the width of each rectangle is $\frac{b-a}{n}$ since we're dividing $[a,b]$ into n equal pieces.

The left-hand endpoint of the i^{th} sub-interval is $i-1$ steps (each of length $\frac{b-a}{n}$) to the right of $x=a$, i.e. it is $x = a + (i-1)\cdot\frac{b-a}{n}$. Thus the i^{th} rectangle has (weighted) area

$$\text{base} \cdot \text{height} = \frac{b-a}{n} \cdot f\left(a + (i-1)\frac{b-a}{n}\right).$$

Textbook just refers to $\frac{b-a}{n}$ as Δx

The n^{th} Left-Hand Rule sum for $\int_a^b f(x)\,dx$

is $$\sum_{i=1}^{n} \frac{b-a}{n} \cdot f\left(a + (i-1)\frac{b-a}{n}\right)$$

$$= \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + (i-1)\frac{b-a}{n}\right).$$

The textbook then <u>defines</u> $\int_a^b f(x)\,dx$

to be
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \frac{(i-1)(b-a)}{n}\right)$$

(this is the Left-Hand Rule).

Caveats: ① You need to be able to compute the limit, at least in principle.

This can be done as long as $f(x)$ is not too discontinuous, <u>ie</u> if has only finitely many jump or removable discontinuities on $[a,b]$ (including at the endpoints).

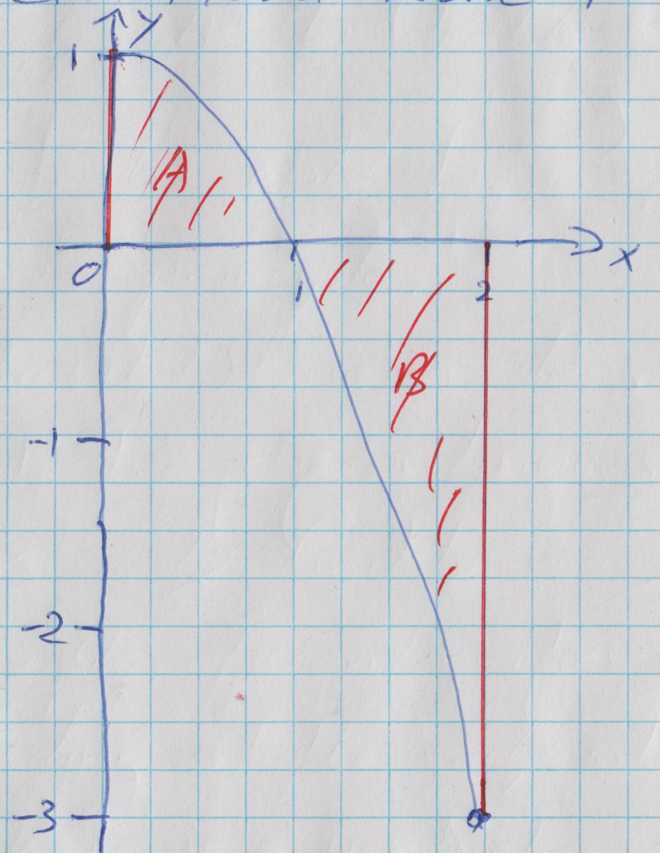
* If there is a vertical asymptote in $[a,b]$ this definition will probably not work...

③ Actually computing the limit, even if it exists, is likely to be pretty annoying, at best.

eg. We'll use the Left-Hand Rule to compute

$$\int_0^2 (1-x^2)\,dx$$

$= A - B$



Plug $f(x) = 1-x^2$ and $a=0$ & $b=2$ into the Left-Hand Rule formula & so:

$$\int_0^2 (1-x^2)\,dx = \lim_{n\to\infty} \frac{2-0}{n}\sum_{i=1}^{n}\left(1-\left[0+(i-1)\frac{2-0}{n}\right]^2\right)$$

$$= \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\left(1-\left[(i-1)\cdot\frac{2}{n}\right]^2\right)$$

$$= \lim_{n\to\infty} \frac{2}{n}\Bigg[\overbrace{\left(1-\left[(1-1)\cdot\frac{2}{n}\right]^2\right)}^{i=1}+\overbrace{\left(1-\left[(2-1)\cdot\frac{2}{n}\right]^2\right)}^{i=2}$$

$$+\overbrace{\left(1-\left[(3-1)\cdot\frac{2}{n}\right]^2\right)}^{i=3}+\cdots$$

$$+\overbrace{\left(1-\left[(n-1)\cdot\frac{2}{n}\right]^2\right)}^{i=n}\Bigg]$$

$$= \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\left(1-\frac{4(i^2-2i+1)}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n}\left(1-\frac{4i^2}{n^2}+\frac{8i}{n^2}-\frac{4}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{2}{n}\Bigg[\left(1-\frac{4\cdot 1^2}{n^2}+\frac{8\cdot 1}{n^2}-\frac{4}{n^2}\right)+\left(1-\frac{4\cdot 2^2}{n^2}+\frac{8\cdot 2}{n^2}-\frac{4}{n^2}\right)$$

$$+\cdots+\left(1-\frac{4\cdot(n-1)^2}{n^2}+\frac{8\cdot(n-1)}{n^2}-\frac{4}{n^2}\right)\Bigg]$$

$$= \lim_{n\to\infty} \frac{2}{n}\Bigg[\left(\sum_{i=1}^{n}1\right)-\left(\sum_{i=1}^{n}\frac{4\cdot i^2}{n^2}\right)+\left(\sum_{i=1}^{n}\frac{8i}{n^2}\right)$$

$$-\left(\sum_{i=1}^{n}\frac{4}{n^2}\right)\Bigg]$$

Fact:

$\sum_{i=1}^{n} i = 1+2+3+\dots+n = \frac{n(n+1)}{2}$

&

$\sum_{i=1}^{n} i^2 = 1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

$$= \lim_{n\to\infty} \frac{2}{n}\left[n - \frac{4}{n^2}\left(\sum_{i=1}^{n} i^2\right) + \frac{8}{n^2}\left(\sum_{i=1}^{n} i\right) - \frac{4}{n^2}\left(\sum_{i=1}^{n} 1\right)\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[n - \frac{4}{n^2}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{8}{n^2}\cdot\frac{n(n+1)}{2} - \frac{4}{n^2}\cdot n\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[n - \frac{2}{3}\cdot\frac{(n+1)(2n+1)}{n} + 4\cdot\frac{n+1}{n} - \frac{4}{n}\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[n - \frac{2}{3}\cdot\frac{2n^2+3n+1}{n} + 4\cdot\frac{n+1}{n} - \frac{4}{n}\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[n - \frac{4}{3}n - 2 - \frac{2}{3n} + 4 + \cancel{\frac{4}{n}} - \cancel{\frac{4}{n}}\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[-\frac{n}{3} + 2 - \frac{2}{3n}\right]$$

$$= \lim_{n\to\infty} \left[-\frac{2}{3} + \frac{4}{n} - \frac{4}{3n^2}\right]$$

$-\frac{2}{3} \to -\frac{2}{3}$, $\frac{4}{n} \to 0$, $\frac{4}{3n^2} \to 0$ as $n\to\infty$

$$= \boxed{-\frac{2}{3}}$$

Check using antiderivatives: Next time!

$$\int_0^2 (1-x^2)\,dx = \left(x - \frac{x^3}{3}\right)\Big|_0^2 = \left(2 - \frac{2^3}{3}\right) - \cancel{\left(0 - \frac{0^3}{3}\right)}$$

$$= 2 - \frac{8}{3} = \frac{6}{3} - \frac{8}{3} = -\frac{2}{3}$$ ✓