Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Winter 2021

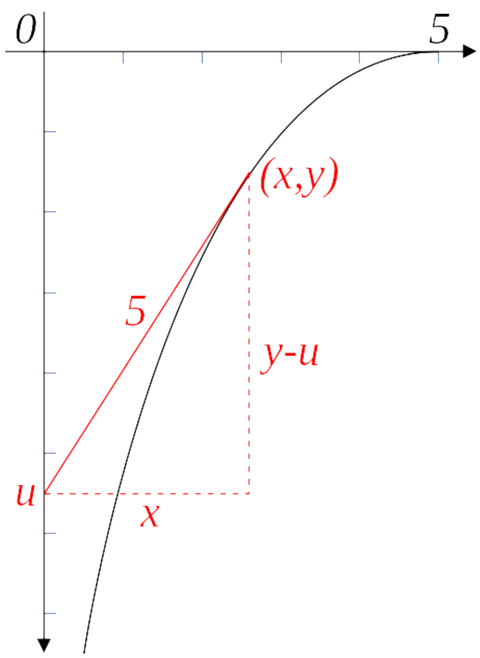
Solutions to Assignment #3
Solving Differential Equations
Due on Friday, 26 February.

Imagine the the Cartesian plane is a flat ice sheet. A sled – which somehow occupies a single point – sits at 5 on the x-axis, with a taut tether attached to a sled dog – which also somehow occupies a single point – at the origin. The dog begins to run straight down the y-axis, pulling on the tether and thus moving the sled. At any given instant, the sled is moving directly towards the dog but, since the tether remains taut and does not stretch, is always 5 units of distance away from the dog.

1. Use the geometry of the situation to set up an equation involving x, y, and $\frac{dy}{dx}$ which must be satisfied by $y = f(x)$, where the graph of $y = f(x)$ is the ~~boat's~~ sled's path. *[6]*

Note: Obviously, we must have $f(5) = 0$. Also, the fact that the sled is always moving directly towards the dog means that the tether is part of the tangent line to the curve $y = f(x)$ at any given instant. The slope of a tangent line is given by – but I must not write too much here ... :-)

SOLUTION. Here is a diagram of the setup, focusing on the instant that the sled is at (x, y) and the dog is $(0, u)$ on the y-axis.



Suppose the sled is at (x, y) at the instant that the dog pulling the sled is at $(0, u)$. The tether is then the hypotenuse of a triangle with vertices at (x, y), $(0, u)$, and (x, u), from which it follows that $(y-u)^2 + x^2 = 5^2 = 25$, so $y - u = \sqrt{25 - x^2}$. The slope of the line the tether runs along at this instant can be then be worked out as follows:

$$m = \frac{\text{rise}}{\text{run}} = \frac{y-u}{x-0} = \frac{\sqrt{25-x^2}}{x} = \frac{1}{x}\sqrt{25-x^2}$$

On the other hand, since the sled moves directly towards the dog at each instant, the tether must be part of the tangent line to the curve followed by the sled at any given instant, so its slope m must also be given by $\frac{dy}{dx}$ at each instant, where $y = f(x)$ gives the curve the sled traverses. Equating these two ways of computing the slope gives us the differential equation

$$\frac{dy}{dx} = \frac{1}{x}\sqrt{25-x^2}$$

with the initial condition that $y = f(0) = 5$ when $x = 0$. ■

2. Use `Maple` or a comparable program to solve this differential equation for $y = f(x)$, with the condition that $f(5) = 0$. *[4]*

Note: If using `Maple`'s worksheet mode, you will want to look up the diff operator and the `dsolve` command.

Solution. Here we are:

```
> dsolve({diff(y(x), x) = sqrt(25 - x^2)/x, y(5) = 0}, y(x))
```

$$y(x) = \sqrt{-x^2+25} - 5\,\mathrm{arctanh}\left(\frac{5}{\sqrt{-x^2+25}}\right) - \frac{5\,I\,\pi}{2} \qquad \textbf{(1)}$$

The sharp-eyed will realize this solution is not entirely satisfactory as the "I" in the final term is the square root of -1. This is probably an artifact of `Maple` trying to solve the given differential equation as generally as possible. A purely real solution (for $0 < x \leq 5$ anyway) would be something like:

$$y = f(x) = \sqrt{25-x^2} - 5\,\mathrm{arcsech}\left(\frac{x}{5}\right) = \sqrt{25-x^2} - 5\ln\left(\frac{5+\sqrt{25-x^2}}{x}\right)$$

You can work this out with the techniques developed in MATH 1120H, though not without some effort.

For those interested, this type of curve is called a *tractrix*. ■