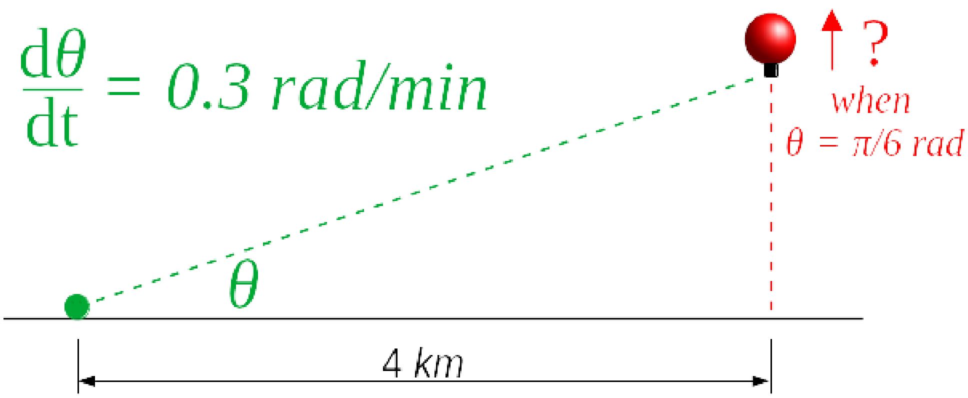
Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Winter 2021

Solution to Quiz #9 *[Corrected 2021-05-11.]*
Tuesday, 23 March.

Show all your work! Simplify where you conveniently can.

1. A hot air balloon rising vertically is tracked by an observer 4 km from the point the balloon lifted off. How fast is the balloon rising at the instant that the observer's line of sight makes an angle of $\frac{\pi}{6}$ rad with the horizontal if this angle is increasing at a rate of 0.3 rad/min at this instant? *[5]*s

SOLUTION. Here's a picture of the setup:



Suppose y is the altitude in kilometres of the balloon above the ground and θ is the angle in radians that the observer's line of sight makes with the horizontal. We are given that the oberver is 4 km from the lift-off point of the balloon and that $\frac{d\theta}{dt} = 0.3\ rad/min$ at the instant that $\theta = \frac{\pi}{6}$ rad. The location of the observer, the lift-off point, and the balloon are the corners of a right triangle, with the right angle at the lift-off point, a base of 4 km, and a height of y km. θ is then the angle the hypotenuse of the triangle (*i.e.* the observer's line of sight) makes with the base. It follows that $\tan(\theta) = \frac{y}{4}$, so $y = 4\tan(\theta)$ and thus

$$\frac{dy}{dt} = \frac{d}{dt} 4\tan(\theta) = 4 \cdot \left[\frac{d}{d\theta}\tan(\theta)\right] \cdot \frac{d\theta}{dt} = 4\sec^2(\theta)\frac{d\theta}{dt}\,.$$

It follows that at the instant that $\theta = \frac{\pi}{6}$ rad we have:

$$\left.\frac{dy}{dt}\right|_{\theta=\pi/6} = 4\sec^2\left(\frac{\pi}{6}\right) \cdot \left.\frac{d\theta}{dt}\right|_{\theta=\pi/6} = 4\left(\frac{2}{\sqrt{3}}\right)^2 \cdot 0.3 = \frac{16}{3} \cdot 0.3 \approx 1.6$$

Thus the balloon is rising at a rate of about 1.6 km/min at the given instant. (This is not a realistic rate of climb for a real hot air balloon, hopefully. 0.3 km/min is already almost certainly unsafe ...) ■

NOTE. Just in case, $\sec^2\left(\frac{\pi}{6}\right) = \frac{1}{\cos^2(\pi/6)} = 1/\left(\sqrt{3}/2\right)^2 = \left(2/\sqrt{3}\right)^2$.