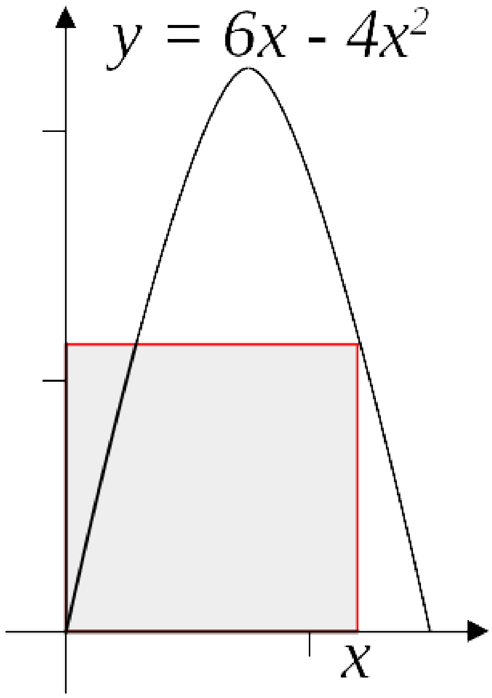
Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
Trent University, Winter 2021

Solution to Quiz #8
Tuesday, 16 March.

Show all your work! Simplify where you conveniently can.

1. A rectangle has its bottom left corner at the origin, its top left corner on the y-axis, its bottom right corner on the x-axis, and its top right corner on the part of the parabola $y = 6x - 4x^2$ that is above the x-axis. Find the maximum area of such a rectangle. *[5]*

Solution. Here is a diagram of the setup:



It's not hard to see that such a rectangle with its top right-hand corner at $(x, y) = \left(x, 6x - 4x^2\right)$ has width $x - 0 = x$ and height $y - 0 = y = 6x - 4x^2$, so it has area $A(x) = x\left(6x - 4x^2\right) = 6x^2 - 4x^3$. For the domain of possible values of x, observe that the part of $y = 6x - 4x^2$ that is above the x-axis is (strictly) between its roots. Since $6x - 4x^2 = 2x(3 - 2x) = 0$ when $x = 0$ or $x = 1.5$, it follows that $0 < x < 1.5$, *i.e.* the relevant domain for x is $(0, 1.5)$.

Technically, we ought to take the limit of $A(x)$ as $x \to 0^+$ and when $x \to 1.5^-$ to figure out what happens at the endpoints of the interval, but since $A(x)$ is defined and continuous at both endpoints (since any polynomial is defined and continuous everywhere) we get away with simply evaluating $A(x)$ at both endpoints: $A(0) = 6 \cdot 0^2 - 4 \cdot 0^3 = 0$ and $A(1.5) = 6 \cdot 1.5^2 - 4 \cdot 1.5^3 = 6 \cdot 2.25 - 4 \cdot 3.375 = 13.5 - 13.5 = 0$.

We next look for critical points in the interval $(0, 1.5)$.

$$A'(x) = \frac{d}{dx}\left(6x^2 - 4x^3\right) = 12x - 12x^2 = 12x(1-x) = 0 \iff x = 0 \text{ or } x = 1$$

$x = 0$ is technically not in the interval $(0, 1.5)$, but as it is one of the endpoints, we have already checked it out above. $x = 1$ is in the interval $(0, 1.5)$, so we check it out too:

$$A(1) = 6 \cdot 1^2 - 4 \cdot 1^3 = 6 - 4 = 2$$

Since $A(1) = 2 > 0 = A(0) = A(1.5)$, it follows that the maximum possible area of a rectangle positioned as specified in the problem is 2. ■