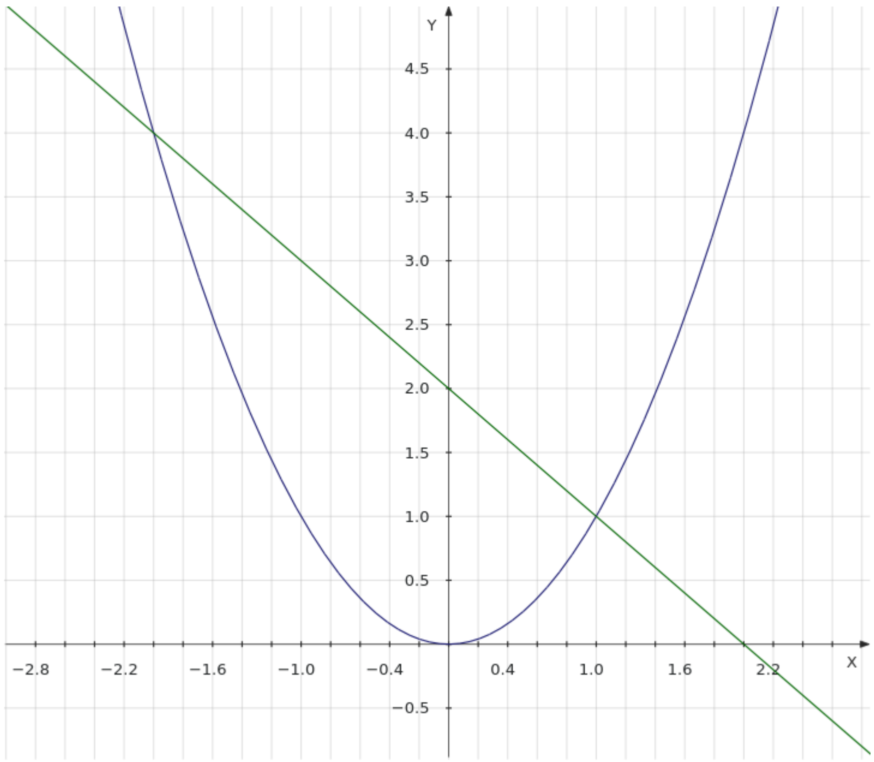
Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Winter 2021

Quiz #11
Tuesday, 6 April.

Show all your work! Simplify where you conveniently can. Compute both of the following integrals.

1. Find the area of the finite region below the line $y = -x + 2$ and above the parabola $y = x^2$. *[2.5]*

SOLUTION. Here's a graph of the functions, as drawn by a program called `kmplot`:



It's easy to tell from the graph that $-2 \leq x \leq 1$ for the region below $y = -x + 2$ and above $y = x^2$. We can verify this – or figure it out to begin with – by noting that the two curves intersect when

$$x^2 = -x + 2 \iff x^2 + x - 2 = 0 \iff (x - 1)(x + 2) = 0 \iff x = 1 \text{ or } x = -2,$$

and that $-x + 2 > x^2$ for $-2 < x < 1$ because $-0 + 2 = 2 > 0 = 0^2$. Since the functions are continuous, we only need to test one point between $x = -2$ and $x = 1$, and $x = 0$ makes for easy arithmetic. :-) It follows that the area of the region is:

$$\int_{-2}^{1} (\text{upper} - \text{lower})\, dx = \int_{-2}^{1} \left((-x+2) - x^2\right)\, dx = \int_{-2}^{1} \left(2 - x - x^2\right)\, dx$$
$$= \left(2x - \frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{-2}^{1}$$
$$= \left(2\cdot 1 - \frac{1^2}{2} - \frac{1^3}{3}\right) - \left(2\cdot(-2) - \frac{(-2)^2}{2} - \frac{(-2)^3}{3}\right)$$
$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - \frac{4}{2} + \frac{8}{3}\right) = \frac{7}{6} - \left(-\frac{10}{3}\right)$$
$$= \frac{7}{6} + \frac{20}{6} = \frac{27}{6} = \frac{9}{2} = 4.5 \qquad \blacksquare$$

2. Find the area of the region between the graphs of $y = xe^{-x}$ and $y = 0$, where $0 \leq x \leq \ln(7)$. *[2.5]*

Solution. Note that $e^{-x} > 0$ for all x, so $xe^{-x} \geq 0$ for $x \geq 0$. It follows that the area of the region is

$$\int_{0}^{\ln(7)} (\text{upper} - \text{lower})\, dx = \int_{0}^{\ln(7)} \left(xe^{-x} - 0\right)\, dx = \int_{0}^{\ln(7)} xe^{-x}\, dx$$

We'll use the substitution $w = -x$, so $x = -w$, $dw = (-1)\, dx$, and $dx = (-1)\, dw$, and change the limits accordingly: $\begin{matrix} x & 0 & \ln(7) \\ w & 0 & -\ln(7) \end{matrix}$

$$= \int_{0}^{-\ln(7)} (-w)e^{w}(-1)\, dw = \int_{0}^{-\ln(7)} we^{w}\, dw$$
$$= -\int_{-\ln(7)}^{0} we^{w}\, dw$$

Now we use integration by parts with $u = w$ and $v' = e^w$, so $u' = 1$ and $v = e^w$.

$$= -\left(we^{w}\big|_{-\ln(7)}^{0} - \int_{-\ln(7)}^{0} e^{w}\, dw \right)$$
$$= -\left(\left[0e^{0}\left(-\ln(7)e^{-\ln(7)}\right)\right] - e^{w}\big|_{-\ln(7)}^{0}\right)$$
$$= -\left(\left[0 - \left(-\frac{\ln(7)}{e^{\ln(7)}}\right)\right] - \left[e^{0} - e^{-\ln(7)}\right]\right)$$
$$= -\left(\frac{\ln(7)}{7} - \left[1 - \frac{1}{e^{\ln(7)}}\right]\right) = -\left(\frac{\ln(7)}{7} - \left[1 - \frac{1}{7}\right]\right)$$
$$= -\left(\frac{\ln(7)}{7} - \frac{6}{7}\right) = -\frac{\ln(7) - 6}{7}$$
$$= \frac{6 - \ln(7)}{7} \approx 0.5792 \qquad \blacksquare$$

Note. *Units? What are units? :-)*