Mathematics 1110H – Calculus I: Limits, derivatives, and Integrals
TRENT UNIVERSITY, Winter 2021

Quiz #10
Tuesday, 30 March.

Available on Blackboard at 12:00 a.m. Tuesday morning.
Due on Blackboard by 11:59 p.m. Tuesday night.
Solutions will be posted on Thursday, 1 April.

Submission: Scanned or photographed solutions are fine, so long as they are legible. Please try to make sure that they are oriented correctly – if they are sideways or upside down, they're rather harder to mark! Submission as a single pdf is strongly preferred, but multiple files and/or other common formats are probably OK in a pinch. Please submit your solutions via Blackboard's Assignments module; if Blackboard does not acknowledge a successful upload, please try again. As a *last* resort, email your solutions to the instructor at: `sbilaniuk@trentu.ca`

Reminder: Per the course outline, *all work submitted for credit must be written up entirely by yourself, giving due credit to all relevant sources of help and information.* For this quiz, you are permitted to use your textbook and all other course material, from this and any other mathematics course(s) you have taken or are taking now, but *you may not use any other sources or aids, nor give or receive any help*, except to ask the instructor to clarify questions and to use a calculator (any that you like).

Show all your work! Simplify where you conveniently can. Compute both of the following integrals.

1. $\displaystyle\int_{1/8}^{1/3} \frac{1}{2x^2\sqrt{1+\frac{1}{x}}}\,dx$ *[2.5]*

SOLUTION. Attempting to simplify matters, we will start off with the substitution $u = \dfrac{1}{x}$. Then $\dfrac{du}{dx} = \dfrac{-1}{x^2}$, so $du = \dfrac{-1}{x^2}\,dx$ and so $\dfrac{1}{x^2}\,dx = (-1)\,du$. We will also change the limits as we go along: $\begin{array}{ccc} x & 1/8 & 1/3 \\ u & 8 & 3 \end{array}$ Then

$$\int_{1/8}^{1/3} \frac{1}{2x^2\sqrt{1+\frac{1}{x}}}\,dx = \int_8^3 \frac{1}{2\sqrt{1+u}}(-1)\,du\,,$$

which we tackle with another substitution, namely $w = 1+u$. Then $\dfrac{dw}{du} = 1$, so $dw = du$, and we change the limits as we go along: $\begin{array}{ccc} u & 8 & 3 \\ w & 9 & 4 \end{array}$ Note that if we had been just a little bit smarter at the beginning, we could have used the substitution $w = 1 + \dfrac{1}{x}$ and saved ourselves doing a second, albeit pretty easy, substitution. Be that as it may, with a little help from the Power Rule and the fact that $\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$, we now have:

$$\begin{aligned}\int_{1/8}^{1/3} \frac{1}{2x^2\sqrt{1+\frac{1}{x}}}\,dx &= \int_8^3 \frac{1}{2\sqrt{1+u}}(-1)\,du = \int_9^4 \frac{-1}{2\sqrt{w}}\,dw = -\frac{1}{2}\int_9^4 w^{-1/2}\,dw \\ &= \frac{1}{2}\int_4^9 w^{-1/2}\,dw = \frac{1}{2}\cdot\left.\frac{w^{1/2}}{1/2}\right|_4^9 = \sqrt{w}\Big|_4^9 = \sqrt{9}-\sqrt{4} \\ &= 3-2 = 1 \qquad \blacksquare\end{aligned}$$

2. $\int \sec^2(\theta)\tan(\theta)\sqrt{1+\sec^2(\theta)}\,dx$ *[2.5]*

Solution. *[Oops! That dx should have been a $d\theta$ ...]* Having possibly learned our lesson from having done two subsitutions instead of just one in the solution above, we will try the substitution $u = 1+\sec^2(\theta)$. Then $\frac{du}{d\theta} = 2\sec(\theta)\cdot\frac{d}{d\theta}\sec(\theta) = 2\sec(\theta)\cdot\sec(\theta)\tan(\theta) = 2\sec^2(\theta)\tan(\theta)$, so $du = 2\sec^2(\theta)\tan(\theta)\,d\theta$ and $\sec^2(\theta)\tan(\theta)\,d\theta = \frac{1}{2}\,du$. It follows that:

$$\begin{aligned}\int \sec^2(\theta)\tan(\theta)\sqrt{1+\sec^2(\theta)}\,d\theta &= \int \sqrt{u}\,\frac{1}{2}\,du = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\cdot\frac{u^{3/2}}{3/2}+C \\ &= \frac{1}{3}u^{3/2}+C = \frac{1}{3}\left(1+\sec^2(\theta)\right)^{3/2}+C \qquad \blacksquare\end{aligned}$$